A Common Fixed Point Theorem under A Contractive Condition of Integral Type

Rakesh Kumar, Arvind Gupta

Abstract- In this paper, we study for some of the fixed point of mapping for a self map on a metric space under a contractive of integral type.

Index Terms- Fixed point, contractive condition, self mapping, lebesgue integral.

INTRODUCTION

The result of this note are inspired by a recent paper of M.S. Khan [4] in 1984 introduced the altering distances and used it for solving fixed points problem in metric spaces. Recently several authors [1, 2, 3, 5, 6,] have used the alternative distance function and obtained some fixed point theorem.

The main aim of this paper is to prove the existence and uniqueness of common fixed point of mapping T, T for a self map on a metric space by using distance function under a contractive condition of integral type.

THEOREM 1.

Let (x,d) be a complete matric space α Є [0,1] $f : x \to x$ a mapping such that for x,y ЄX.

$$\int_0^{d(fx,fy)} \psi(t)dt \le \alpha \int_0^{d(x,y)} \psi(t)dt$$

where $\psi : [0,+\infty) \to [0,+\infty)$ is a lebesgue integral mapping which is sum able non negative and such that $\int_0^{\varepsilon} \psi(t)dt > 0$ for each $\varepsilon > 0$, then f has a unique fixed point ZЄ X, such that for each x ЄX, $\lim_{n\to\infty} f^n(x) = Z$.

THEOREM 2.

Let (x,d) be complete metric space and let $T : x \to x$ be self- mapping satisfying the inequality

$\varphi\{d(Tx,Ty)\} \le \psi_1\{d(x,y) - \psi_2(dx,y)\}$,

where $\psi_1, \psi_2, \varphi : [0,\infty) \to [0,\infty)$ are continues and monotone non decreasing functions with $\psi_1(t) = \psi_2(t) = \varphi(t) = 0$, if and only if t =0. Then T has a unique fixed point.

DEFINITION:-

Let ψ_n denote the set of all variables,

i. ψ is continues.

ii. ψ is monotone increasing in all the variable.

iii. $\psi(t_1, t_2, t_3 ...., t_n) = 0$ if and only if $t_1 = t_2 = t_3 = ... = t_n = 0$.

We define $\varphi(x) = \psi(x, x, x, ...)$ for $x \in [0, \infty)$.

Clear $\varphi(x) = 0$

Example of ψ are

$\psi(t_1, t_2, t_3, ..., t_n) = k \max\{t_1 = t_2 = t_3 = ...t_n\}$ for k>0 ……….. (1)

$\psi(t_1, t_2, t_3, ..., t_n) = t_1^{a_1} + t_2^{a_2} + ... + t_n^{a_n}$,

where $a_1, a_2, a_3, ... \ge 1$ (2)

Vahid Reza Hosseini [6] has proved the following theorem,

THEOREM 3.:-

Let (x,d) be a complete metric space and S,T:X→X such that

$$\int_0^{\varphi_1\{d(sx,ty)\}} \psi(t)dt \le \int_0^{\psi_1\{d(x,y),d(sx,x),d(Ty,y)\}} \psi(t)dt - \int_0^{\psi_2\{d(x,y),d(sx,x),d(Ty,y)\}} \psi(t)\ dt$$

for all x,y $\in$ X where $\psi_1, \psi_2 \in \psi_3$, $\varphi_1 = \psi(x, x, x...)$ and $x \in [0, \infty)$,

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where $\psi : R^+ \to R^+$ is a lebesgue integral mapping which is sum able non negative and such that $\int_0^{\varepsilon} \psi(t)dt > 0$ for each $\varepsilon > 0$, then S,T have a unique common fixed point inX.

Now we prove the following theorem:-

MAIN RESULT

Let(x,d) be a complete metric space and S,T,P:X→ X satisfying the following condition

(i)

$$\int_0^{\varphi_1\{d(SPx,TPy)\}} \psi(t)dt \le \int_0^{\psi_1\{d(x,y),d(x,SPx),d(y,TPy),d(SPx,TPy)\}} \psi(t)dt$$

$$-\int_0^{\psi_2\{d(x,y),d(x,SPx),d(y,TPy),d(SPx,TPy)\}} \psi(t)dt \qquad (3)$$

For all x,y ε X where $\psi_1, \psi_2 \in \psi_3$ and $\varphi = \psi(x, x, x...), x \in [0, \infty)$.

(ii) One of three mapping S, T and P is continuous.

(iii) Where $\psi : R^+ \to R^+$ is a lebesgue integral mapping which is sum able non negative and such that $\int_0^{\varepsilon} \psi(t)dt > 0$ for each $\varepsilon > 0$. Further assume that SP=PS or TP=PT, then S,T and P have common unique fixed point in X.

Proof:- Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{x_n\}$ as

$$x_{2n+1} = SPx_{2n}, n = 0,1,2,3,...... \text{ and}$$

$$x_{2n+2} = TPx_{2n+1}, n = 0,1,2,3,......$$

Let $a_n = d(x_n, x_{n+1})$ ……………………. (4)

Putting $x_n = x_{2n}$ and $y = x_{2n+1}$ for all n = 1,2,3,……………

We get from (3)

$$\int_0^{\varphi_1\{d(SPx_{2n},TPx_{2n+1})\}} \psi(t)dt \le \int_0^{\psi_1\{d(x_{2n},x_{2n+1}),d(x_{2n},SP_{2n}),d(x_{2n+1},TPx_{2n+1}),d(SPx_{2n},TPx_{2n+1})\}} \psi(t)dt$$

$$-\int_0^{\psi_2\{d(x_{2n},x_{2n+1}),d(x_{2n},SP_{2n}),d(x_{2n+1},TPx_{2n+1}),d(SPx_{2n},TPx_{2n+1})\}} \psi(t)dt$$

$$\int_0^{\varphi_1\{d(x_{2n+1},TPx_{2n+2})\}} \psi(t)dt \le$$

$$\int_0^{\psi_1\{d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),d(x_{2n+1},x_{2n+2})\}} \psi(t)dt$$

$$-\int_0^{\psi_2\{d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),d(x_{2n+1},x_{2n+2})\}} \psi(t)dt$$

From (4)

$$\int_0^{\varphi_1 a_{2n+1}} \psi(t)dt \le \int_0^{\psi_1(a_{2n},a_{2n},a_{2n+1},a_{2n+1})} \psi(t)dt - \int_0^{\psi_2(a_{2n},a_{2n},a_{2n+1},a_{2n+1})} \psi(t)dt \qquad (5)$$

If $a_{2n+1} > a_{2n}$,then

$$\int_0^{\varphi_1 a_{2n+1}} \psi(t)dt \le \int_0^{\psi_1(a_{2n+1},a_{2n+1},a_{2n+1},a_{2n+1})} \psi(t)dt < \int_0^{\varphi_1 a_{2n+1}} \psi(t)dt \qquad (6)$$

Which is contracdition. Hence $a_{2n+1} \le a_{2n}, n = 0,1,2,3,....$

Similarly by putting $x = x_{2n}$,and $y = x_{2n-1}$ in (3)

We can show that $a_{2n+2} \le a_{2n+1}$, n=0,1,2,3,…….

Thus $a_{n+1} \le a_n$,n=0,1,2,3…….

So that $\{a_n\}$ is a decreasing sequence of non negative real number and hence convergent to some $a \in R$.

…………………….(7)

From (5) and (6) for all n=1,2,3,………. obtain

$$\int_0^{\varphi_1 a_{n+1}} \psi(t)dt \le \int_0^{\psi_1 a_n} \psi(t)dt - \int_0^{\psi_2 a_n} \psi(t)dt$$

Then $\int_0^{\varphi_2 a_{n+1}} \psi(t)dt \le \int_0^{\varphi_1 a_n} \psi(t)dt - \int_0^{\varphi_1 a_{n+1}} \psi(t)dt$

Summing up we obtain

$$\sum_{n=0}^{\infty} \int_0^{\varphi_2 a_{n+1}} \psi(t)dt \le \int_0^{\varphi_1 a_0} \psi(t)dt \le \infty$$

Which implies $\varphi_2(a) \to 0$ which imply that $n \to \infty$

Again from (7) $\{a_n\}$ is convergent and $a_n \to a$ as $n \to \infty$ since φ is continuous, we obtain $\varphi(a) = 0$,Which implies that a=0, that is $a = d(x_{n+1}, x_n) \to 0$ as $n \to \infty$

…………………………… (8)

We next prove that $\{x_n\}$ is a Cauchy sequence , in view of (7) it is sufficient to prove that $\{x_{2r}\}_{r=1}^{\infty} \subset \{x_n\}$ is Cauchy sequence. If $\{x_{2r}\}_{r=1}^{\infty}$ is not Cauchy sequence of natural number {2mk},{2nk} such that nk > mk, $d(x_{2mk}, x_{2nk}) \ge \varepsilon$

$$d(x_{2mk}, x_{2nk-1}) < \varepsilon \qquad (9)$$

Then by (8)

$$\varepsilon < d(x_{2mk}, x_{2nk}) \le d(x_{2mk}, x_{2nk-1}) + d(x_{2mk}, x_{2nk-1})$$
$$< \varepsilon + d(x_{2nk}, x_{2nk-1})$$

Making $k \to \infty$ in the above inequality by virtue of (7) We obtain

$$\lim_{n\to\infty} d(x_{2mk}, x_{2nk}) = \varepsilon \qquad (10)$$

For all k=1,2,3……..

$$d(x_{2nk+1}, x_{2mk}) \le d(x_{2nk+1}, x_{2nk}) + d(x_{2nk}, x_{2mk}) \qquad (11)$$

Also for all k=1,2,3,……..

$$d(x_{2nk}, x_{2mk}) \le d(x_{2nk}, x_{2nk+1}) + d(x_{2nk+1}, x_{2mk}) \qquad (12)$$

Making $k \to \infty$ in (9)and (10) repectively, by using (7) and (8), we have

$\lim_{k\to\infty} d(x_{2nk+1}, x_{2m}) \le \varepsilon$ and $\varepsilon \le \lim_{k\to\infty} d(x_{2nk+1}, x_{2m})$

$\lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)}) = \varepsilon$ for all k= 1,2,3,……
………..(13)

$$d(x_{2n(k)}, x_{2m(k)-1}) \le d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)-1})$$

$$d(x_{2n(k)}, x_{2m(k)}) \le d(x_{2n(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)})$$

Taking $k \to \infty$ in the above two inequalities and using (7) and (8) we obtain

$$\lim_{k\to\infty} d(x_{2nk}, x_{2m-1}) = \varepsilon$$

Putting $x = x_{2nk}$ and $y = x_{2mk-1}$ in (3), for all k=1,2,3,……
We obtain

$$\int_0^{\varphi_1 d(x_{2nk+1}, x_{2mk})} \psi(t)dt \le$$
$$\int_0^{\psi_1 d(x_{2nk}, x_{2mk-1}), d(x_{2nk}, x_{2mk-1}), d(x_{2mk-1}, x_{2mk}), d(x_{2nk+1}, x_{2mk})} \psi(t)dt$$
$$- \int_0^{\psi_2 d(x_{2nk}, x_{2mk-1}), d(x_{2nk}, x_{2mk-1}), d(x_{2mk-1}, x_{2mk}), d(x_{2nk+1}, x_{2mk})} \psi(t)dt$$

Making $k \to \infty$ in the above inequality and taking into account the continuity of and by (7),(8),(9) . WE have,

$\int_0^{\varphi_1(\varepsilon)} \psi(t)dt \le \int_0^{\psi_1(\varepsilon,0,0,0)} \psi(t)dt - \int_0^{\psi_2(\varepsilon,0,0,0)} \psi(t)dt$,the
n

$$\varphi_1(\varepsilon) \le \psi_1(\varepsilon,0,0,0) - \psi_2(\varepsilon,0,0,0) < \varphi_1(\varepsilon)$$

This is due the fact that ψ_1 is monotone increasing in its variables and by property that $\psi(x, y, z) = 0$ if and only if x=y=z=0
The above inequality give a contradiction so that $\varepsilon = 0$.
This establishes sequence and hence convergence in (X, d).
Let $x_n \to z$ as $n \to \infty$
Putting $x = x_{2n}$ and x=y in(3) for all n=1,2,3,………

$$\int_0^{\varphi_1 d(x_{2n+1}, TPz)} \psi(t)dt \le \int_0^{\psi_1 d(x_{2n}, z), d(x_{2n}, x_{2n+1}), d(z, TPz), d(x_{2n+1}, TPz)} \psi(t)dt$$
$$- \int_0^{\psi_2 d(x_{2n}, z), d(x_{2n}, x_{2n+1}), d(z, TPz), d(x_{2n+1}, TPz)} \psi(t)dt$$

Making $n \to \infty$ in the above inequality, by using (8) and (12) and continuity of ψ_1 and ψ_2, we obtain

$$\int_0^{\varphi_1 d(z,TPz)} \psi(t)dt \le \int_0^{\psi_1(0,0,0,d(z,TPz)} \psi(t)dt$$
$$- \int_0^{\psi_2(0,0,0,d(z,TPz)} \psi(t)dt$$

If $d(z,TPz) \ne 0$, then using property that ψ_1 and ψ_2 are monotone increasing and $\psi_2(x, y, z) = 0$ if and only if x=y=z=0,we obtain

$$\int_0^{\varphi_1 d(z,TPz)} \psi(t)dt \le \int_0^{\varphi_1 d(z,TPz)} \psi(t)dt$$

Which is contradictions. Hence ,we obtain d(z,TPz)=0,or z = TPz in an exactly similarly way to prove z = SPz
Thus SPz = z= TPz (14)

Also fallows that z is the common fixed point of SP and TP.
Suppose S is continuous, then
$S^2Px_{2n} = Sz$, $Sx_{2n} = Sz$ .If SP = PS.
Then putting $x = Sx_{2n}, y = x_{2n+1}$ in (3),

$$\int_0^{\varphi_1 d(S^2Px_{2n}, TPx_{2n+1})} \psi(t)dt \le$$
$$\int_0^{\psi_1 d(Sx_{2n}, x_{2n+1}), d(Sx_{2n}, S^2Px_{2n}), d(x_{2n+1}, TPx_{2n+1}), d(S^2Px_{2n}, TPx_{2n+1})} \psi(t)dt$$

$$-\int_0^{\psi_2 d(Sx_{2n},x_{2n+1}),d(Sx_{2n},S^2Px_{2n}),d(x_{2n+1},TPx_{2n+1}),d(S^2Px_{2n},TPx_{2n+1})}\psi(t)dt$$

$$\Rightarrow \int_0^{\varphi_1 d(Sz,z)}\psi(t)dt \le \int_0^{\psi_1 d(Sz,z),d(Sz,Sz),d(z,z),d(Sz,z)}\psi(t)dt - \int_0^{\psi_2 d(Sz,z),d(Sz,Sz),d(z,z),d(Sz,z)}\psi(t)dt$$

$$\Rightarrow \int_0^{\psi_1 d(Sz,z),0,0,d(Sz,z)}\psi(t)dt - \int_0^{\psi_2 d(Sz,z),0,0,d(Sz,z)}\psi(t)dt < \int_0^{\varphi_1 d(Sz,z)}\psi(t)dt$$

It is a contradiction, if $Sz \neq z$,hence Sz = z.
Hence by (14) Tz = z = Sz.
Similarly if TP = PT, then also Pz = z = Sz = Tz.
Hence this z is a common fixed point of S,T,P.
Thus the theorem is proved

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Dr. K. QURESHI
ADDL. DIRECTOR (Rtd.)
DEPARTMENT OF HIGHER EDUCATION
GOVT.OF M.P.

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