

A Creative Review on Integer Additive Set-Valued Graphs

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Abstract— For a non-empty ground set X , finite or infinite, the set-valuation or set-labeling of a given graph G is an injective function $f: V(G) \rightarrow \mathcal{P}(X)$, with induced edge-function $f^*: E(G) \rightarrow \mathcal{P}(X)$ is defined by $f^+(uv) = f(u) * f(v); uv \in E(G)$, where $\mathcal{P}(X)$ is the power set of X and $*$ is a pre-defined binary operation on sets. A set-labeling f of a graph G is called a set-indexer if the induced function f^* is also injective. An integer additive set-labeling is defined as an injective function $f: V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^*: E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective, where \mathbb{N}_0 is the set of all non-negative integers, $\mathcal{P}(\mathbb{N}_0)$ its power set $f(u) + f(v)$ is the sumset of the set-labels $f(u)$ and $f(v)$ of the vertices u and v in G . In this paper, we critically and creatively review the concepts and properties of integer additive set-valued graphs.

Index Terms— Set-labeling, set-indexer, integer additive set-labeling, integer additive set-indexer.

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1 INTRODUCTION

FOR all terms and definitions, not defined specifically in this paper, we refer to [23]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

The researches on graph labeling problems commenced with the introduction of the concept of number valuations of graphs in [29]. Since then, the studies on graph labeling have contributed significantly to the researches in graph theory and associated fields. Graph labeling problems have numerous theoretical and practical applications. Many types of graph labelings are surveyed and listed in [16].

Motivated from certain problems related to social interactions and social networks, Acharya introduced, in [1], the notion of set-valuation of graphs analogous to the number valuations of graphs. For a non-empty ground set X , finite or infinite, the set-valuation or set-labeling of a given graph G is an injective function $f: V(G) \rightarrow \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of the set X .

Also, in [1], a set-indexer of a graph G is defined as an injective set-valued function $f: V(G) \rightarrow \mathcal{P}(X)$ such that the function $f^*: E(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$ defined by $f^*(uv) = f(u) * f(v)$ for every $uv \in E(G)$ is also injective, where $\mathcal{P}(X)$ is the set of all subsets of X and $*$ is a binary operation on sets.

Taking the symmetric difference of two sets as the operation between two set-labels of the vertices of G , the following theorem was proved in [1].

Theorem 1.1. [1] *Every graph has a set-indexer.*

2. ON INTEGER ADDITIVE SET-LABELED GRAPHS

Let A and B be two non-empty sets. The *sumset* of A and B is denoted by $A + B$ and is defined as $A + B = \{a + b: a \in A, b \in B\}$. If a set C is the sumset of two sets A and B , then A and B are said to be the *summands* of C . Note that for any set A , we can write $A = A + \{0\}$. Here, A and $\{0\}$ are said to be the *non-trivial summands* of the set A . The sumsets and summands are non-trivial unless mentioned otherwise.

For any positive integer n , an *n-integral multiple* of a set A , denoted by $n.A$ and is defined as $n.A = \{na: a \in A\}$. We denote the cardinality of a set A by $|A|$. Hence, $2A = A + A$ and $2.A = \{2a: a \in A\}$ and $|2A| > |A|$ where as $|2.A| = |A|$.

In this work, we review the studies on the graphs which admit a particular type of set-labeling called integer additive set-labeling, defined in terms of the sumsets of given sets of non-negative integers.

The notion of integer additive set-labeling of a graph was introduced as follows, using the concept of sumsets of two sets of non-negative integers, as follows.

Definition 2.1. Let \mathbb{N}_0 be the set of all non-negative integers and let $\mathcal{P}(\mathbb{N}_0)$ be its power set. A function $f: V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$, whose associated function $f^*: E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ is defined by $f^+(uv) = f(u) + f(v)$, is said to be an *integer additive set-labeling* if it is injective. A graph G which admits an integer additive set-labeling is called an *integer additive set-labeled graph* or *integer additive set-valued graph*.

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The notion of integer additive set-indexers of graphs was first appeared in [17] as follows.

Definition 2.2. An *integer additive set-indexer* (IASI) of a graph G is defined as an injective function $f: V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+: E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v) = \{a + b: a \in f(u), b \in f(v)\}$ is also injective. A graph G which admits an IASI is called an integer additive set-indexed graph.

In this discussion, we denote the associated function of an IASL f , defined over the edge set of G by f^+ .

Definition 2.3. An IASL (or IASI) is said to be k -uniform if $|f^+(e)| = k$ for all $e \in E(G)$. That is, a connected graph G is said to have a k -uniform IASL if all of its edges have the same set-indexing number k .

If either $f(u)$ or $f(v)$ is countably infinite, then their sumset will also be countably infinite and hence the study of the cardinality of the sumset $f(u) + f(v)$ becomes countably infinite. Hence, we restrict our discussion to finite sets of non-negative integers.

Certain studies about integer additive set-indexed graphs have been done in [5], [18], [32] and [31]. The following are the important terms and definitions made in these papers.

The cardinality of the set-label of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element. If the set-labels of all vertices of G have the same cardinality, then the vertex set $V(G)$ is said to be *uniformly set-indexed*. A graph is said to admit (k, l) -completely uniform IASI (or IASL) f if f is a k -uniform IASL (or IASI) and $V(G)$ is l -uniformly set-indexed.

Analogous to Theorem 1, we have proved the following theorem.

Theorem 2.4. Every graph has an integer additive set-labeling.

The proof follows from the fact that the sumset of two sets of non-negative integers is again a set of non-negative integers.

The following theorem was proved in [31] by choosing the set-labels of the vertices of a given graph G in a suitable manner.

Theorem 2.5. [31] Every graph has an integer additive set-indexer.

3. IASL OF CERTAIN GRAPH OPERATIONS

The sumsets of different pairs of sets non-negative integers need not be distinct in all cases. But we can establish the existence of an IASI for any given graph G if we choose the set-labels of the vertices of G suitably in such a way that no two edges of G have the same set-indexing number.

We know that if A and B are two sets of non-negative integers, then their sumset $A + B$ is also a set of non-negative integers. That is, if $A, B \subset \mathbb{N}_0$, then $A + B \subset \mathbb{N}_0$. This property leads us to the following results on certain operations and products of IASL graphs.

First, we consider the union of two IASL graphs.

Theorem 3.1. The union of two IASL (or IASI) graphs also admits (induced) IASL (or IASI).

Proof. Let G_1 and G_2 be two graphs that admit the IASLs f_1 and f_2 respectively, chosen in such a way that f_1 and f_2 are the same for the common vertices of G_1 and G_2 , if any. Then, define a function f on $G_1 \cup G_2$, such that $f = f_1$ for all vertices in G_1 and $f = f_2$ for all vertices in G_2 such that the associated function f^+ of f is defined as $f^+(uv) = f_1(u) + f_1(v)$ if $uv \in E(G_1)$ and $f^+(uv) = f_2(u) + f_2(v)$ if $uv \in E(G_2)$. Clearly, f is an IASL of $G_1 \cup G_2$. \square

Next, in the following theorem, we check whether the join of two IASL graphs is an IASL graph.

Theorem 3.2. The join of two IASL (or IASI) graphs also admits (induced) IASL (or IASI).

Proof. Let G_1 and G_2 be two graphs that admit the IASLs f_1 and f_2 respectively. Consider the join $G = G_1 + G_2$. Define a function f on G in such a way that $f = f_1$ for all vertices in G_1 and $f = f_2$ for all vertices in G_2 and the associated function $f^+: E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ of f is defined as $f^+(uv) = f_1(u) + f_1(v)$ if $uv \in E(G_1)$ and $f^+(uv) = f_2(u) + f_2(v)$ if $uv \in E(G_2)$. Clearly, f is an IASL of $G_1 \cup G_2$. \square

The following theorem is on the admissibility of an IASL by the complement of an IASL-graph.

Theorem 3.3. The complement of an IASL graph is also an IASL graph.

Proof. The proof follows from the fact that a graph G and its complement \bar{G} have the same set of vertices and hence have the same set of set-labels. That is, an IASL f defined on a graph G is an IASL of the complement of G also. \square

Now, recall the definitions of the certain fundamental products of two graphs given in [22].

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. Then, the *Cartesian product* of G_1 and G_2 , denoted by $G_1 \square G_2$, is the graph with vertex set $V_1 \times V_2$ defined as follows. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two points in $V_1 \times V_2$. Then, u and v are adjacent in $G_1 \square G_2$ whenever $[u_1 = v_1 \text{ and } u_2 \text{ is adjacent to } v_2]$ or $[u_2 = v_2 \text{ and } u_1 \text{ is adjacent to } v_1]$.

The *direct product* of two graphs G_1 and G_2 , is the graph whose

vertex set is $V(G_1) \times V(G_2)$ and for which the vertices (u, v) and (u', v') are adjacent if $uu' \in E(G_1)$ and $vv' \in E(G_2)$. The direct product of G_1 and G_2 is denoted by $G_1 \times G_2$.

The *strong product* of two graphs G_1 and G_2 is the graph, denoted by $G_1 \boxtimes G_2$, whose vertex set is $V(G_1) \times V(G_2)$ and for which the vertices (u, v) and (u', v') are adjacent if $[uu' \in E(G_1) \text{ and } v = v']$ or $[u = u' \text{ and } vv' \in E(G_2)]$ or $[uu' \in E(G_1) \text{ and } vv' \in E(G_2)]$.

The *lexicographic product* or *composition* (see [24]) of two graphs G_1 and G_2 is the graph, denoted by $G_1 \circ G_2$, is the graph whose vertex set $V(G_1) \times V(G_2)$ and for two vertices (u, v) and (u', v') are adjacent if $[uu' \in E(G_1)]$ or $[u = u' \text{ and } vv' \in E(G_2)]$.

Now we proceed to verify whether these products of two IASL graphs admit IASLs.

Theorem 3.4. *The Cartesian product of two IASL graphs also admits an IASL.*

Proof. Let u_1, u_2, \dots, u_m be the vertices of G_1 and v_1, v_2, \dots, v_n be the vertices of G_2 . Also let f_1, f_2 be the IASLs defined on G_1 and G_2 respectively. Now, define a function f on $G_1 \square G_2$ such that $f(u_i, v_j) = f_1(u_i) + f_2(v_j)$ where the associated function $f^+ : (E(G_1 \square G_2)) \rightarrow \mathcal{P}(\mathbb{N}_0)$ is defined by $f^+((u_i, v_j), (u_r, v_s)) = f(u_i, v_j) + f(u_r, v_s)$. Therefore, f is an IASL on $G_1 \square G_2$. \square

Using the definition of f for the vertices and the associated function f^+ for the edges, we can establish the existence of IASLs for the other fundamental graph products also as stated in the following results.

Theorem 3.5. *The direct product of two IASL graphs also admits an IASL.*

Theorem 3.6. *The strong product of two IASL graphs also admits an IASL.*

Theorem 3.7. *The lexicographic product of two IASL graphs is also an IASL graph.*

Next, we consider certain products of graphs in which we take finite number copies of any one of the two graphs and attach these copies to the vertices of the other graph using certain rules.

The most popular one among such graph products, is the corona of two graphs which is defined in [15] as follows.

The *corona* of two graphs G_1 and G_2 , denoted by $G_1 \odot G_2$, is the graph obtained by taking $|V(G_1)|$ copies of G_2 and then joining the i -th point of G_1 to every point of the i -th copy of G_2 .

Another commonly used graph product in various literature, is the rooted product of two graphs, which is defined in [19] as follows.

The *rooted product* of a graph G_1 on n_1 vertices and rooted graph G_2 on n_2 vertices, denoted by $G_1 \circ G_2$, is defined as the graph obtained by taking n_1 copies of G_2 , and for every vertex v_i of G_1 and identifying v_i with the root node of the i -th copy of G_2 .

The admissibility of an IASL by these products of two IASL graphs is established in the following theorem.

Theorem 3.8. *The corona of two IASL graphs also admits an IASL.*

Proof. Let u_1, u_2, \dots, u_m be the vertices of G_1 and let v_1, v_2, \dots, v_n be the vertices of G_2 . For $1 \leq i \leq m$ and $1 \leq j \leq n$, let v_{ij} be the j -th vertex of the i -th copy of G_2 . Now, label the vertex v_{ij} of i -th copy G_2 corresponding to vertex v_j of G_2 by the set $f_i(v_{ij}) = i \cdot f(v_j)$. Let the set-label of an edge in G be the sumset of the set-labels of its end vertices. Clearly, this labeling is an IASL on $G_1 \odot G_2$. \square

Theorem 3.9. *The rooted product of two IASL graphs is also an IASL graph.*

Proof. In the rooted product $G_1 \circ G_2$, for $1 \leq i \leq |V(G_1)|$, the roots of the i -th copy of G_2 is identified with the i -th vertex of G_1 . Label these identified vertices by the corresponding set-labels of G_1 . All other vertices in the different copies of G_2 by distinct integral multiples of the set-labels of corresponding vertices of G_2 . Also, let the set-labels of edges in $G_1 \circ G_2$ be the sumset of the set-labels of their end vertices. Clearly, this set-labeling is an IASL on $G_1 \circ G_2$. \square

4. IASL of Certain Associated Graphs

An IASL of a graph G induces an IASL to certain other graphs associated to G . The induction property of a IASL of a graph G to various associated graphs of G is studied in [31]. The major concepts and findings in this paper are mentioned in this section. More over, we establish some new results in this area.

An obvious result about an IASL f of a given graph G is its hereditary nature. The following theorem shows that an IASL of a given graph induces an IASL on any subgraph of G .

Proposition 4.1. [31] *If a given graph G admits an IASL (or IASI) f , then any subgraph H of G also admits an IASL f^* , the restriction of f to $V(H)$. That is, the existence of an IASL for a graph G is a hereditary property.*

If we replace an element (a vertex or an edge) of G by another element (a vertex or an edge) which is not in G to form an associated graph structure, say G' , it is customary that the common elements of the graph G and the associated graph G' preserve the same set-labels and the newly introduced elements assume the same set-label of the corresponding deleted elements of G . That is, the IASL defined on G induces an IASL for the graph G' . The IASL, thus defined for the newly formed graph G' is called an induced IASL of G' .

This induction property of integer additive set-labeling holds not only for the subgraphs of an IASI graph G , but also for certain other graph structures associated with G such as minors, topological reductions etc.

In the following results, we consider only the IASL of the associated graph which are induced from the IASL of the graph G concerned.

By an *edge contraction* in G , we mean an edge, say e , is removed and its two incident vertices, u and v , are merged into a new vertex w , where the edges incident to w each correspond to an edge incident to either u or v .

The existence of an IASI for the graph obtained by contracting finite edges of an IASL graph G is established in the following theorem.

Proposition 4.2. [31] *Let G be an IASI graph and let e be an edge of G . Then, $G \circ e$ admits an IASI.*

An undirected graph H is called a minor of a given graph G if H can be formed from G by deleting edges and vertices and by contracting edges.

Proposition 4.3. *Let G be an IASI graph and let H be a minor of G . Then, H admits an IASI.*

The proof of this result is an immediate consequence of Proposition 4.1 and 4.2.

Let G be a connected graph and let v be a vertex of G with $d(v) = 2$. Then, v is adjacent to two vertices u and w in G . If u and w are non-adjacent vertices in G , then delete v from G and add the edge uw to $G - \{v\}$. This operation is called an elementary topological reduction on G . Then, we have

Proposition 4.4. [31] *Let G be a graph which admits an IASI. Then any graph G' , obtained by applying finite number of elementary topological reductions on G , admits an IASI.*

A *subdivision* or an *expansion* of a graph is the reverse operation of an elementary topological reduction. Under this operation, we introduce a new vertex, say w , to an edge uv in G . Thus, the vertex w replaces the edge uv and two edges uw and wv are introduced to $G - uv \cup \{w\}$. Two graphs G and G' are said to be homeomorphic if there is a graph isomorphism from some subdivision of G to some subdivision of G' .

The following theorem establishes the existence of an induced IASL for a graph G' which is homeomorphic to a given IASL graph G .

Proposition 4.5. *Let G be an IASL (or IASI) graph. Then, any subdivision G' of G admits an (induced) IASL (or IASI).*

Proof. Let f be an IASL defined on a given graph G and let u, v

be two adjacent vertices in G . Introduce a new vertex w on the edge uv so that uv is subdivided to two edges uw and wv . Let $G' = [G - (uv)] \cup \{uw, vw\}$. Define a function $g: V(G') \rightarrow \mathcal{P}(\mathbb{N}_0)$ such that $g(v) = f(v)$ if $v \in V(G)$ and $g(w) = f^+(uv)$. Clearly, g is an IASL on G' . This result can be extended to a graph obtained by finite number of subdivisions on G . Therefore, any subdivision of an IASL graph admits an (induced) IASL. \square

Invoking Proposition 5, the admissibility of an IASL by a graph that is homeomorphic to a given IASL graph.

Proposition 4.6. *Let G be an IASL (or IASI) graph and let H be a graph that is homeomorphic to G . Then, H admits an IASL (or IASI).*

Proof. Assume that the graphs G and H are homeomorphic graphs. Let G' and H' be the isomorphic subdivisions of G and H respectively. Let f be an IASL defined on G . By Proposition 5, the subdivision G' admits an IASL, say f' induced by f . Since $G' \cong H'$, define a function $g: V(H') \rightarrow \mathcal{P}(\mathbb{N}_0)$ by $g(v') = f(v)$, where v' is the vertex in H' corresponding to the vertex v of G' . That is, g assigns the same set-labels of the vertices of G' to the corresponding vertices of H' . Therefore, $g(V(H')) = f(V(G'))$. Since H can be considered as a graph obtained by applying a finite number of topological reductions on H' , by Proposition 4, g induces an IASL on H . Therefore, H is an IASL graph. \square

The existence of integer additive set-labelings for certain other graphs associated to the given graph G , which are not obtained by replacing some elements of G by certain other elements not in G , have been established in [31].

An important graph of this kind which is associated to a given graph G is the line graph of G which is defined as follows.

For a given graph G , its *line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G incident on a common vertex in G . The following theorem establishes the existence of IASI by the line graph of a graph.

Theorem 4.7. [31] *If a graph G admits IASI, say f , then its line graph $L(G)$ also admits an (induced) IASI.*

Another graph that is associated to a given graph is its total graph which is defined as follows.

The *total graph* of a graph G is the graph, denoted by $T(G)$, is the graph having the property that a one-to one correspondence can be defined between its points and the elements (vertices and edges) of G such that two points of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent (either if both elements are edges or if both elements are vertices) or they are incident (if one element is an edge and the other is a vertex).

The following theorem establishes the existence of an IASI for

the total graph of a graph.

Theorem 8. [31] If a graph G admits IASI, say f , then its total graph $T(G)$ also admits an (induced) IASI.

So far, we discussed about the integer additive set-labeling of graphs with respect to the countably infinite set \mathbb{N}_0 . Can we choose a finite set X as the ground set for labeling the vertices of a graph G ? This is possible only when the ground set X has sufficiently large cardinality. Hence, the study about the cardinality of the ground set X arises much interest.

The minimum cardinality of the ground set that is required for set-labeling a graph G is called the set-indexing number of G . The following theorem determines the set-indexing number of a given graph G .

Theorem 4.9. [31] Let X be a finite set of non-negative integers and let $f: V(G) \rightarrow 2^X - \{\emptyset\}$ be an IASI on G , which has n vertices. Then, X has at least $\lceil \log_2(n+1) \rceil$ elements.

The following theorem is special case of the above theorem when the vertices of G have the same set-indexing number.

Theorem 4.10. [31] Let X be a finite set of non-negative integers and let $f: V(G) \rightarrow 2^X - \{\emptyset\}$ be an IASI on G , which has n vertices, such that $V(G)$ is l -uniformly set-indexed. Then, $n = C(|X|, l)$.

We have discussed the cardinality of the ground set X for labeling the vertices of a given graph so that it admits an IASL. The cardinality of the set-labels of the elements of G is also worth studying. Some studies in this area are reviewed in the following section.

5. Cardinality of the Set-Labels of IASL Graphs

Certain studies on set-indexing numbers of the elements of the IASI-graphs have been done in [18], [32] and [31]. The set-theoretic foundations of the IASLs, established in these studies, are reviewed in this section.

Let A and B be two non-empty sets of non-negative integers. Then the ordered pairs (a, b) and (c, d) in $A \times B$ are said to be compatible if $a + b = c + d$. If (a, b) and (c, d) are compatible, then we write $(a, b) \sim (c, d)$. Clearly, \sim is an equivalence relation.

A compatible class with respect to an integer k is the subset of $A \times B$ defined by $\{(a, b) \in A \times B: a + b = k\}$ and is denoted by C_k .

All the compatibility classes need not have the same number of elements but can not exceed a certain number. It can be noted that exactly one representative element of each compatibility class C_k of $f(u) \times f(v)$ contributes an element to the set-label of the corresponding edge uv and all other $r_k - 1$ elements are neglected and we call these elements as neglected elements of the class C_k . The number of neglected elements in

the set-label of an edge uv with respect to the set $f(u) \times f(v)$ is called the neglecting number of that edge.

The compatibility classes which contain the maximum possible number of elements are called saturated classes. The compatibility class that contains maximum elements is called a maximal compatibility class. Hence, we have

Proposition 5.1. [31] The cardinality of a saturated class in $A \times B$ is n , where $n = \min(|A|, |B|)$.

The number of distinct compatibility classes in $A \times B$ is called the compatibility index of the pair of sets (A, B) and is denoted by $\mathfrak{U}_{(A,B)}$. Hence, we have the following lemma.

Lemma 5.2. Let f be an IASI of a graph G and u, v be two vertices of G . The set-indexing number of an edge $e = uv$ of G is given by $|f^+(e)| = |f(u)| + |f(v)| = \mathfrak{U}_{(f(u), f(v))}$.

The following result establishes the bounds for the set-indexing number of the edges of an IASI graph, in terms of the set-indexing numbers of their end vertices.

Lemma 5.3. [18] For an IASI f of a graph G , $\max(|f(u)|, |f(v)|) \leq |f^+(uv)| = |f(u) + f(v)| \leq |f(u)||f(v)|$, where $u, v \in V(G)$.

Theorem 5.4. Let f be an IASI of a graph G and let u and v be two adjacent vertices of G . Let $|f(u)| = m$ and $|f(v)| = n$. Then, $|f^+(uv)| = mn - r$, where r is the number of neglected elements in $f(u) \times f(v)$.

An interesting question that arises in this context is about a necessary and sufficient condition for an IASL of a given graph G to be a uniform IASL of G . The following result provides a platform for finding the conditions for a given graph G to admit a uniform IASL.

Proposition 5.5. Two adjacent edges e_i and e_j of a graph G have the same set-indexing number if and only if the set-indexing number of the common vertex of these edges is the quotient when the difference of the neglecting numbers of the edges is divided by the differences of set-indexing numbers of the end vertices that are not common to these edges.

Proof. Let $e_i = uv_i$ and $e_j = uv_j$ be two adjacent edges in the given graph G . Let f be an IASL defined on G . Also, let $|f(u)| = m$, $|f(v_i)| = n_i$ and $|f(v_j)| = n_j$. Then, the set-indexing number of e_i is $mn_i - r_i$ and that of e_j is $mn_j - r_j$ where r_i and r_j are neglecting numbers of e_i and e_j respectively.

Assume that e_i and e_j have the same set-indexing number. Then, $mn_i - r_i = mn_j - r_j$. That is, $m = (r_i - r_j)/(n_i - n_j)$.

Conversely, assume that the set-indexing number of the common vertex of these edges is the quotient when the difference of the neglecting numbers of the edges is divided by the dif-

ferences of set-indexing numbers of the end vertices that are not common to these edges. That is, $m = (r_i - r_j)/(n_i - n_j)$. Hence, $mn_i - r_i = mn_j - r_j$.

Therefore, the edges have the same set-indexing number. \square

Due to the above proposition, we establish a necessary and sufficient condition for an IASL of a given graph to be a uniform IASL of G .

Theorem 5.6. *Let G be a graph that admits an IASL f . Then, f is a uniform IASL of G if and only if the set-indexing number of the common vertex of any two adjacent edges is the ratio of the difference between their neglecting numbers to the difference between the set-indexing numbers of their distinct end vertices.*

In the following theorem, we prove a necessary and sufficient condition for an IASL of a graph G , which assigns uniform set-labels to the vertices of G , to be a uniform IASL of G .

Theorem 5.7. *Let the graph G admits an IASL f under which all vertices of G are uniformly set-labeled. Then, f is a uniform IASL if and only if all edges of G have the same neglecting number.*

Proof. Let $V(G)$ be l -uniformly set-labeled. Then, for any vertex u in G , $|f(u)| = l$ and $f^+(e_i)f^+(uv) = l^2 - r_i$. Let f be a uniform IASL of G . Then, for any two adjacent edges $e_i = uv_i$ and $e_j = uv_j$ of G , we have $|f^+(e_i)| = |f^+(e_j)|$. That is, $l^2 - r_i = l^2 - r_j$ and hence $r_i = r_j$.

Conversely, assume that all the edges of G have the same neglecting number, say r . Therefore, for any edge $e = uv$ of G , $|f^+(e)| = |f(u)||f(v)| - r$. Since the graph $V(G)$ is l -uniformly set-labeled, $|f(v)| = l$ for all $v \in V(G)$. Therefore, $|f^+(e)| = l^2 - r$ for all edges $e \in E(G)$. That is, f is a uniform IASL. \square

A necessary and sufficient condition for a graph to have a 2-uniform IASI is proved in [5] as follows.

Theorem 5.8. [5] *A graph G admits a 2-uniform IASI if and only if G is bipartite.*

In view of Lemma 3 the following definitions are introduced in [18] and [32].

Definition 5.9. [18] A *weak IASI* is an IASI f whose associated function is defined as $|f^+(uv)| = \max(|f(u)|, |f(v)|)$ for all $u, v \in V(G)$. A graph which admits a weak IASI may be called a *weak IASI graph*. A weak IASI is said to be *weakly uniform IASI* if $|f^+(uv)| = k$, for all $u, v \in V(G)$ and for some positive integer k .

Definition 5.10. [32] An IASI f is called a *strong IASI* if $|f^+(uv)| = |f(u)||f(v)|$ for all $u, v \in V(G)$. A graph which admits a strong IASI may be called a *strong IASI graph*. A strong IASI is said to be *strongly uniform IASI* if $|f^+(uv)| = k$, for all $u, v \in V(G)$ and for some positive integer k .

Form the above definitions it follows that if $f: V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ is a weak (or strong) IASI defined on a graph G , then for each adjacent pair of vertices u and v of G , each compatibility class of the pair of set-labels $f(u)$ and $f(v)$ is a trivial class.

Let G be an IASI graph. Then, the following result for G was proved in [18].

Lemma 5.11. [18] *Let $f: V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ be an IASI defined on a graph G . Let u and v be any two adjacent vertices in G . Then, $|f^+(uv)| = \max(|f(u)|, |f(v)|)$ if and only if either $|f(u)| = 1$ or $|f(v)| = 1$.*

This lemma leads us to the following result.

Theorem 5.12. [31] *An IASI f of a graph G is a weak IASI of G if and only if at least one end vertex of any edge in G has the set-indexing number 1.*

The following theorem describes a necessary and sufficient condition for a graph to admit a weakly uniform IASI.

Theorem 5.13. [31] *A connected graph G admits a weakly uniform IASI f if and only if G is a bipartite graph.*

The necessary and sufficient condition for the sumset of two sets to have maximum cardinality is established in the following result.

Lemma 5.14. [32] *Let A, B be two non-empty subsets of \mathbb{N}_0 . Let D_A and D_B be the sets of all differences between two elements in the sets A and B respectively. Then, $|A + B| = |A| \cdot |B|$ if and only if D_A and D_B are disjoint.*

The sets D_A of all differences between two elements of set A is called the *difference set* of A .

In view of the above lemma, a necessary and sufficient condition for an IASI to be a strong IASI is established in [32] as follows.

Theorem 5.15. [32] *An IASI f of a graph G is a strong IASI of G if and only if the difference sets of the set-labels of any two adjacent vertices in G are disjoint sets.*

The following theorem describes a necessary and sufficient condition for a graph to admit a strongly uniform IASI.

Theorem 5.16. [32] *A connected graph G admits a strongly k -uniform IASI f if and only if G is a bipartite graph or f is a (k, l) -completely uniform IASI of G , where $k = l^2$.*

6. Conclusion

So far, we have reviewed the studies about integer additive set-labelings and integer additive set-indexers of certain graphs and their characteristics. In this paper, we also propose some new results in this area. Certain problems regarding the admissibility of integer additive set-indexers by various other

graph classes are still open.

More properties and characteristics of different types of IASLs and IASIs, both uniform and non-uniform, are yet to be investigated. There are some more open problems regarding necessary and sufficient conditions for various graphs and graph classes to admit certain IASIs. All these facts show that there is a wide scope for further researches in this area.

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REFERENCES

- [1] B. D. Acharya, *Set-Valuations and Their Applications*, MRI Lecture notes in Applied Mathematics, The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad, 1983.
- [2] B. D. Acharya, *Set-Indexers of a Graph and Set-Graceful Graphs*, Bulletin of Allahabad Mathematical Society, (2001), 1-23.
- [3] B. D. Acharya and K. A. Germina, *Strongly Indexable Graphs: Some New Perspectives*, *Advanced Modeling and Optimisation*, (1)(2013), 3-22.
- [4] S. Alexander, *The Mathematical Coloring Book*, Springer-Verlag, 2008.
- [5] T. M. K. Anandavally, *A Characterisation of 2-Uniform IASI Graphs*, *International Journal of Contemporary Mathematical Sciences*, (10)(2013), 459-462.
- [6] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1989.
- [7] M. Behzad, *Total graphs*, in *Graph Theory and Applications*, (Editors: Y. Alavi, D. R. Lick, A. T. White), pp. 21-23, Springer, 1972.
- [8] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan Press, London, 1976.
- [9] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, 2008.
- [10] A. Brandstädt, V. B. Le and J. P. Spinrad, *Graph Classes: A Survey*, SIAM, Philadelphia, 1999.
- [11] D. M. Burton, *Elementary Number Theory*, Tata McGraw-Hill Inc., New Delhi, 2007.
- [12] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, McGraw-Hill Inc., 2005.
- [13] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, CRC Press, 2000.
- [14] N. Deo, *Graph theory With Application to Engineering and Computer Science*, Prentice Hall of India, 1974.
- [15] R. Frucht and F. Harary *On the Corona of Two Graphs*, *Aequationes Mathematicae*, (3)(1970), 322-325.
- [16] J. A. Gallian, *A Dynamic Survey of Graph Labelling*, *The Electronic Journal of Combinatorics (DS #16)*, 2013.
- [17] K. A. Germina and T. M. K. Anandavally, *Integer Additive Set-Indexers of a Graph: Sum Square Graphs*, *Journal of Combinatorics, Information and System Sciences*, (2-4)(2012), 345-358.
- [18] K. A. Germina, N. K. Sudev, *On Weakly Uniform Integer Additive Set-Indexers of Graphs*, *International Mathematical Forum*, (37-40)(2013), 1827-1834.
- [19] C. D. Godsil and B. D. McKay, *A New Graph Product and its Spectrum*, *Bulletin of Australian Mathematical Society*, (1978), 21-28.
- [20] J. T. Gross and J. Yellen, *Graph Theory and its Applications*, CRC Press, 2006.
- [21] G. Hahn and G. Sabidussi, *Graph Symmetry: Algebraic Methods and Applications*, NATO Advanced Science Institute Series-C 497, Springer, 1997.
- [22] R. Hammack, W. Imrich and S. Klavzar *Handbook of Product graphs*, CRC Press, (2011).
- [23] F. Harary, *Graph Theory*, Addison-Wesley, 1969.
- [24] W. Imrich, S. Klavzar, *Product Graphs: Structure and Recognition*, Wiley, 2000.
- [25] K. D. Joshi, *Applied Discrete Structures*, New Age International, 2003.
- [26] M. B. Nathanson, *Additive Number Theory, Inverse Problems and Geometry of Sumsets*, Springer, New York, 1996.
- [27] M. B. Nathanson, *Elementary Methods in Number Theory*, Springer-Verlag, New York, 2000.
- [28] M. B. Nathanson, *Additive Number Theory: The Classical Bases*, Springer-Verlag, New York, 1996.
- [29] A. Rosa, *On certain valuation of the vertices of a graph*, in *Theory of Graphs*, Gordon and Breach, 1967.
- [30] K. H. Rosen, *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, 2000.
- [31] N. K. Sudev and K. A. Germina, *On Integer Additive Set-Indexers of Graphs*, *International Journal of Mathematical Sciences & Engineering Applications*, (II)(2014), 11-22.
- [32] N. K. Sudev and K. A. Germina, *Some New Results on Strong Integer Additive Set-Indexers*, *Discrete Mathematics, Algorithms and Algorithms*, (1)(2015), 11-pages.
- [33] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics*, CRC press, 2011.
- [34] D. B. West, *Introduction to Graph Theory*, Pearson Education Inc., 2001.
- [35] Information System on Graph Classes and their Inclusions, <http://www.graphclasses.org/smallgraphs>.