A Description of Non Linear Schrödinger Equation by Modified Exp-Function Method

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Abstract— In this paper, the modified exp-function method is used to obtain wave solutions of the Non Linear Schrödinger (NLS) equation and to construct a model for the nonlinear evolution of instability. As a result, some new types of exact wave solutions are obtained which include kink wave solutions, periodic wave solution, and solitary wave solutions. Obtained results clearly indicate the reliability and efficiency of the proposed modified exp-function method.

Keywords: Non Linear Schrödinger equation, exact wave solution, modified exp-function method

1. Introduction

The investigation of exact solutions of nonlinear wave equations plays an important role in the study of nonlinear physical phenomena. Recently, many effective methods for obtaining exact solutions of nonlinear wave equations have been proposed, such as bäcklund transformation method [1], homogeneous balance method [2,3], bifurcation method [4], Hirotas bilinear method [5], the hyperbolic tangent function expansion method [6,7], Jacobi elliptic function expansion method [8,9], F-expansion method[10-12] and so on. He and Wu [13] developed the exp-function method to seek the solitary, periodic and compaction like solutions of differential equations. This widely used, effective and simple method is extended here as modified exp-function expansion method. The purpose of this paper is to find exact wave solutions of the Non Linear Schrödinger equation by the new method.

The Non Linear Schrödinger (NLS) equation arises as the envelope of a dispersive wave system which is almost monochromic and weakly nonlinear. The NLS equation has found numerous applications in physics, in the theory of deep-water waves [14], as well as a model for the non-linear pulse propagation in fibers [15,16] in the Heisenberg model of magnetics etc. We consider the Non Linear Schrödinger equation which describes the evolution of the amplitude of a quasi-monochromatic wave propagating in a weakly (cubic) nonlinear medium

$$iE_t + \alpha E_{xx} + \beta |E|^2 E = 0, \qquad (1)$$

Here *E (x, t)* is the slowly varying envelope of high-frequency field.

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2. Modified exp-function method

The exp-function method was first proposed by He and Wu to solve differential equations [13] and it was systematically studied in [17-19]. In this paper, we introduce the modified exp-function method whose main procedure is as follows. For a general nonlinear partial differential equation in the form of

$$H\left(E, E_{t,} E_{x,} E_{tt,} E_{xx} E_{tx} \ldots\right) = 0 \qquad (2)$$

where $E = E$ *(x, t)* is the solution of the Equation (2), we use transformations $\xi = x - ct$ (3)

Where *c* is a constant, to obtain the operators

$$\frac{\partial}{\partial t}(\ ) = -c\frac{\partial}{\partial \xi}(\ ), \frac{\partial}{\partial x}(\ ) = \frac{\partial}{\partial \xi}(\ ), \frac{\partial^2}{\partial t^2}(\ ) = c^2\frac{\partial^2}{\partial \xi^2}(\ )$$

The transformations (3) are used to change the nonlinear partial differential equation (1) to the nonlinear ordinary differential equation $H_1(E, E', E'', E''', ..) = 0$ (4)

Where the prime denotes the derivation with respect to ξ.

Let $E = v + s$ (5)

Where s is constants then equation (4) becomes

$$H_2\left(v, v', v'', v''' \ldots\ldots\right) = 0 \qquad (6)$$

Assume that the solution of equation (4) can be expressed in the following form

$$E(\xi) = \frac{\sum_{i=-n}^{n} \chi_i g^i}{\sum_{i=-n}^{n} \xi_i g^i} = \frac{\sum_{i=0}^{2n} b_i g^i}{\sum_{i=0}^{2n} a_i g^i} \qquad (7)$$

Where $g = e^{-\kappa\xi}$, which is the solution of the homogeneous linear equation corresponding to equation (6), a_i, b_i are unknown to be further determined and *n* can be determined by homogeneous balance principle. Substituting Equation (7) into

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Equation (4), we get polynomial in *g*. If the coefficient of g^i be zero, then on solving the equation set, a_i, b_i can be determined.

3. Solutions of nonlinear Schrödinger equation

We use the wave transformations

$$E(x,t)=u(\xi)\exp[i(kx-\omega t)]$$

$$\xi = p(x-2\alpha kt)$$

equation (1) can be rewritten as

$$\alpha p^2 u'' + u(\omega-\alpha k^2)+\beta u^3 = 0, \quad (8)$$

Let $u = v(\xi)+s$

Where s is constants then equation (8) becomes

$$\alpha p^2 v'' + v(\omega-\alpha k^2)+\beta(v^3+3v^2 s+3vs^2) + s\left[(\omega-\alpha k^2)+\beta s^2\right]=0 \quad (9)$$

Let $s\left[(\omega-\alpha k^2)+\beta s^2\right]=0$

Then $s=0$ and $s=\pm\sqrt{-\left[\frac{(\omega-\alpha k^2)}{\beta}\right]}$

According to homogeneous balance principle, we get n = 1

Case 1 $s=0$ Then $u(\xi)=v(\xi)$,

The solution of the linear equation corresponding to equation (9) is $g=e^{-\kappa\xi}$, $\kappa=\sqrt{\frac{(\alpha k^2-\omega)}{\alpha p^2}}$ (10)

Thus, we look for the solution of (3) in the form

$$u(\xi)=\frac{\sum_{i=0}^{2n} b_i g^i}{\sum_{i=0}^{2n} a_i g^i}=\frac{b_0+b_1 g+b_2 g^2}{a_0+a_1 g+a_2 g^2}$$

Substituting equation (11) and equation (11) into equation (9) yields a set of algebraic equations for g^i (i = 0, 1...6). Let the coefficient of these terms g^i are zero, that yields a set of over-determined algebraic equations.

$$\begin{cases}\beta b_0^2+(\omega-\alpha k^2)b_0 a_0^2=0,\\ 3(\omega-\alpha k^2)b_0 a_1 a_0+3\beta b_0^2 b_1=0\\ 3\beta b_0^2 b_2+3\beta b_1^2 b_0-3(\omega-\alpha k^2)b_2 a_0^2+3(\omega-\alpha k^2)b_1 a_1 a_0\\ +6(\omega-\alpha k^2)b_0 a_2 a_0=0\\ (\omega-\alpha k^2)b_1 a_1^2-(\omega-\alpha k^2)a_2 a_1+6\beta b_0 b_1 b_2\\ +8(\omega-\alpha k^2)b_1 a_0 a_2-(\omega-\alpha k^2)b_2 a_1 a_0+\beta b_1^3=0\\ 3(\omega-\alpha k^2)b_1 a_0 a_2+3\beta b_1^2 b_2-3(\omega-\alpha k^2)b_0 a_2^2\\ +6(\omega-\alpha k^2)b_2 a_2 a_0+3\beta b_2^2 b_0=0\\ 3\beta b_2^2 b_1+3(\omega-\alpha k^2)b_2 a_1 a_2=0\\ (\omega-\alpha k^2)b_2 a_2^2+\beta b_2^3=0\end{cases} \quad (11)$$

Solving the system of algebraic equations

$$a_0=-\frac{\beta b_1^2}{8(\omega-\alpha k^2)a_2} \quad a_1=0,\ b_0=0,\ b_2=0 \quad (12)$$

Where $a_2\neq 0$, b_1 is arbitrary constant

Then
$$u(\xi)=\frac{b_1 g}{-\frac{\beta b_1^2}{8(\omega-\alpha k^2)a_2}+a_2 g^2}$$

Then the solution of Non Linear Schrödinger equation

$$E(x,t)=\left[\frac{b_1}{-\frac{\beta b_1^2}{8(\omega-\alpha k^2)a_2}g^{-1}+a_2 g}\right]\exp[i(kx-\omega t)] \quad (13)$$

Where $g=e^{-\kappa\xi}$, $\kappa=\sqrt{\frac{2(\omega-\alpha k^2)}{\alpha p^2}}$ (14)

Case 2 $s=\pm\sqrt{-\left[\frac{(\omega-\alpha k^2)}{\beta}\right]}$ Then $u(\xi)=s+v(\xi)$,

The solution of the linear equation corresponding to equation (8) is $g=e^{-\kappa\xi}$, $\kappa=\sqrt{\frac{2(\omega-\alpha k^2)}{\alpha p^2}}$

Thus, we look for the solution of (8) in the form

$$v(\xi)=\frac{\sum_{i=0}^{2n} b_i g^i}{\sum_{i=0}^{2n} a_i g^i}=\frac{b_0+b_1 g+b_2 g^2}{a_0+a_1 g+a_2 g^2} \quad (15)$$

Substituting Equation (14) and Equation (15) into Equation (8) yields a set of algebraic equations for g^i *(i = 0, 1...6).* If the coefficient of these terms g^i to be zero then the set of over-determined algebraic equations are:

$$\begin{cases}\beta b_0^3+(\omega-\alpha k^2)b_0 a_0^2=0,\\ 3\beta b_0^2 b_1+3(\omega-\alpha k^2)a_0^2 b_1=0\\ 9(\omega-\alpha k^2)a_0^2 b_2+3\beta b_0^2 b_2+3\beta b_0 b_1^2+3(\omega-\alpha k^2)a_1^2 b_0\\ -6(\omega-\alpha k^2)b_0 a_2 a_0=0\end{cases}$$

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$$\begin{cases} 8\left(\omega-\alpha k^{2}\right)a_{1}a_{2}b_{0}+\left(\omega-\alpha k^{2}\right)b_{1}a_{1}^{2} \\ +6\beta b_{0}b_{1}b_{2}+8\left(\omega-\alpha k^{2}\right)b_{2}a_{0}a_{1}-10\left(\omega-\alpha k^{2}\right)b_{1}a_{2}a_{0}+\beta b_{1}^{3}=0 \\ 3\left(\omega-\alpha k^{2}\right)b_{2}a_{1}^{2}+3\beta b_{1}^{2}b_{2}-6\left(\omega-\alpha k^{2}\right)b_{2}a_{0}a_{2} \\ +9\left(\omega-\alpha k^{2}\right)b_{0}a_{2}^{2}+3\beta b_{2}^{2}b_{0}=0 \\ 3\beta b_{2}^{2}b_{1}+3\left(\omega-\alpha k^{2}\right)b_{1}a_{2}^{2}=0 \\ \left(\omega-\alpha k^{2}\right)b_{2}a_{2}^{2}+\beta b_{2}^{3}=0 \end{cases}$$

Solving the system of algebraic equations by use of Maple, we obtain

$$a_{0}=\pm\frac{\left(\omega-\alpha k^{2}\right)+\beta b_{1}^{2}}{4b_{2}\sqrt{-\beta\left(\omega-\alpha k^{2}\right)}} \qquad a_{2}=\pm b_{2}\sqrt{-\frac{\beta}{\left(\omega-\alpha k^{2}\right)}},$$

$$b_{0}=\frac{\left(\omega-\alpha k^{2}\right)a_{0}^{2}+\beta b_{1}^{2}}{4b_{2}\beta},$$

Where $b_{2}\neq 0$, b_{1} a_{1} is arbitrary constant, Then

$$v(\xi)=\frac{\dfrac{\left(\omega-\alpha k^{2}\right)a_{0}^{2}+\beta b_{1}^{2}}{4b_{2}\beta}+b_{1}g+b_{2}g^{2}}{\pm\dfrac{\left(\omega-\alpha k^{2}\right)+\beta b_{1}^{2}}{4b_{2}\sqrt{-\beta\left(\omega-\alpha k^{2}\right)}}+a_{1}g\pm b_{2}\sqrt{-\dfrac{\beta}{\left(\omega-\alpha k^{2}\right)}}g^{2}}$$

Then the solution of nonlinear Schrödinger equation

$$E(x,t)=\left[\frac{\dfrac{\left(\omega-\alpha k^{2}\right)a_{0}^{2}+\beta b_{1}^{2}}{4b_{2}\beta}+b_{1}g+b_{2}g^{2}}{\pm\dfrac{\left(\omega-\alpha k^{2}\right)+\beta b_{1}^{2}}{4b_{2}\sqrt{-\beta\left(\omega-\alpha k^{2}\right)}}+a_{1}g\pm b_{2}\sqrt{-\dfrac{\beta}{\left(\omega-\alpha k^{2}\right)}}g^{2}} \pm\sqrt{-\left[\frac{\left(\omega-\alpha k^{2}\right)}{\beta}\right]}\right]\exp\left[i\left(kx-\omega t\right)\right]$$

4. Conclusions

The solution of Non Linear Schrödinger equation is derived using the modified exp-function method and it is shown that the new method is powerful and straightforward for nonlinear differential equations. It can be deduced that this method can be applied to other kinds of nonlinear problems also.

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