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A Simple Design Approach for the Fuzzy Control of Nonlinear Dynamic System via Lyapunov Stability Theorem

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Abstract— This paper presents the control design and the stability analysis of the continuous time fuzzy control system. The nonlinear dynamic system can exactly be represented by T-S fuzzy model which consists of number of linear subsystem. Based on the lyapunov stability theorem, the stability conditions are obtained in the term of LMI and the feedback gain matrices of the each linear state feedback controller is obtained by the method of pole placement.

Index Terms— Takagi-Sugeno (T-S) fuzzy model, Linear Matrix Inequality (LMI), Fuzzy State Feedback Controller

1 Introduction

Fuzzy system are developed and applied in various fields of application. Fuzzy logic control has been applied successfully to an automatic train operation, an automatic container crane operation, elevator control, nuclear reactor control, and motor control. In general, fuzzy control system can be classified as Mamdani type and Takagi-Sugeno (T-S) type. The Mamdani type fuzzy control system is well recognized and received by the society. The T-S type fuzzy system mainly focuses on the modeling aspect.

There are some works in literature that are mainly concerned with the stability analysis of T-S fuzzy model. The existence of a proper T-S fuzzy model is first assumed. Tanaka and Sugeno [1] [2] [3] showed that finding a common symmetric positive definite matrix P could show the stability of a T-S fuzzy model.

One of the most important concepts concerning the properties of control systems is stability. Stability analysis of fuzzy control systems has been difficult because fuzzy systems are essentially nonlinear systems. There are several approaches to control of a nonlinear system. A typical approach is the feedback stabilization of nonlinear system where a linear feedback control is designed for the linearization of nonlinear systems where about a nominal operating point. This approach, however, generally only renders a local result. Other approaches such as feedback linearization are rather involved and tend to result in rather complicated controllers.

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In this paper, we consider a non-local approach [4], which is conceptually simple and straightforward. Linear feedback control techniques can be utilized as in the case of feedback stabilization. The procedure is as follows: First the nonlinear plant is represented by a Takagi-Sugeno type fuzzy model. In this type of fuzzy model, local dynamic state space regions are represented by linear models.

The control design is carried out base on the fuzzy model which uses the same membership function of given fuzzy model. The idea is that for each linear model, a linear feedback control designed is carried out base on the fuzzy model.

This paper is organized as follows. In section 2, T-S fuzzy model and fuzzy state feedback controller are introduced. In section 3, stability condition for the closed loop fuzzy control system is presented. In section 4, a numerical example is presented to illustrate the proposed methodology. In section 5, a conclusion will be drawn.

2 Problem formulation

Consider the nonlinear dynamic system $\dot{x}(t) = f(x(t), u(t), t)$. This nonlinear dynamic system can be exactly represented by T-S fuzzy model proposed by Takagi and Sugeno fuzzy model [3] with model rule.

2.1 T-S fuzzy model

This method is simple and natural. The system dynamics is captured by a set of fuzzy implications, which characterize local relations in the state space. The main feature of a Takagi-Sugeno fuzzy model [4] [5] is to express the local dynamics of each fuzzy implication (rule) by a linear system model. The fuzzy model suggested by Takagi & Sugeno is of the following form:

2.2 Continuous Time Fuzzy System

Model rule i

IF $z_1(t) is\ M_{i1}\ and\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots. z_p(t) is\ M_{ip}$;

$for\ i = 1,2,3 \ldots\ldots. r$

THEN

$\dot{x}(t)$=$A_i x(t) + B_i u(t)$

where $\boldsymbol{M_{ij}}$ is the fuzzy set; $j = 1,2,3 \ldots\ldots\ldots. r$;

$x(t) \epsilon R^n$ is the state vector;

$u(t) \in R^m$ is the **control input**;

$z_1(t), z_2(t)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots. z_p(t)$ are the premise variable and **r** is the no. of fuzzy **IF-THEN** rules.

By fuzzy blending, the overall fuzzy model is inferred as:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i \big(z(t)\big)\big(A_i x(t) + B_i u(t)\big) \quad (1)$$

Where $\mu_i = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))}$;

$w_i\big(z(t)\big) = \prod_{j=1}^{p} M_{ij}\left(z_j(t)\right)$;

$w_i\big(z(t)\big)\ is\ the\ degree\ of\ membership\ of\ z_j(t) in\ M_{ij}$

Here $w_i\big(z(t)\big) \geq 0, for\ i = 1,2,3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. r$

and $\sum_{i=1}^{r} w_i\big(z(t)\big) > 0$ for all t.

Therefore, $\mu_i\big(z(t)\big) \geq 0$ for $i = 1,2,3 \ldots\ldots. r$ and $\sum_{i=1}^{r} \mu_i\big(z(t)\big) = 1$.

2.3 Fuzzy state feedback controller

Control Rule i

IF $z_1(t) is\ M_{i1}\ and\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots. z_p(t) is\ M_{ip}$;

$for\ i = 1,2,3 \ldots\ldots. r$

THEN

$\boldsymbol{u}(t) = -\boldsymbol{F_i x}(t)$

where $\boldsymbol{M_{ij}}$ is the fuzzy set; $j = 1,2,3 \ldots\ldots\ldots. r$; $x(t) \epsilon R^n$ is the state vector;

The fuzzy control rules have a linear controller (state feedback laws in this case) in the consequent parts.

The output of Fuzzy controller is given by:

$$\boldsymbol{u}(t) = -\sum_{i=1}^{r} \mu_i\,(z(t))\boldsymbol{F_i x}(t) \quad (2)$$

Put the value of (1) into (2), we obtain (3) as given below

$$\boldsymbol{\dot{x}(t)}=\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{j=1}^{r} \mu_j \big(z(t)\big)(A_i - B_i F_j)\boldsymbol{x(t)} \quad (3)$$

Let $H_{ii} = (A_i - B_i F_i)$

The control design problem is to find local feedback gains $\boldsymbol{F_i}$ such that conditions **(4)** and **(5)** in Theorem 1 are satisfied. Using the notation of quadratic stability, we can also think of the control design problem as finding F_i, s such that the closed-loop system (3) is quadratically stable.

3 Stability Analysis of Fuzzy Control System in the Sense of Lyapunov Stability Theorem

A sufficient condition, derived by Tanaka and Sugeno [6] which guarantees the stability of a fuzzy system, is obtained in term of Lyapunov's direct method.

Theorem 1: *The equilibrium of a fuzzy control system (3) is asymptotically stable in the large if there exists a common positive definite matrix P such that the following two conditions are satisfied.*

$$\{\boldsymbol{A_i} - \boldsymbol{B_i F_i}\}^{\boldsymbol{T}} \boldsymbol{P} + \boldsymbol{P}\{\boldsymbol{A_i} - \boldsymbol{B_i F_i}\} < 0, \ \ i = 1,2, \ldots\ldots. r \quad (4)$$

$$\boldsymbol{H_{ij}}^{\boldsymbol{T}} \boldsymbol{P} + \boldsymbol{P H_{ij}} < 0, i < j \leq \boldsymbol{r} \quad s.t \quad \mu_i \cap \mu_j \neq \emptyset \quad (5)$$

$where\ \ \boldsymbol{H_{ij}} = \frac{\{\boldsymbol{A_i} - \boldsymbol{B_i F_j}\} + \{\boldsymbol{A_j} - \boldsymbol{B_j F_i}\}}{2}$

The condition (4) & (5) are in the form of LMI, which can be solved by the LMI tool box in matlab [7].

Proof: From the equation (3) we have

$$\boldsymbol{\dot{x}(t)}=\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{j=1}^{r} \mu_j \big(z(t)\big)(A_i - B_i F_j)\boldsymbol{x(t)}$$

Let $H_{ii} = (A_i - B_i F_i)$

Above can also be written as:

$$\boldsymbol{\dot{x}(t)}=\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{j=1}^{r} \mu_j \big(z(t)\big) H_{ii}\boldsymbol{x(t)} + 2\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{i<j} \mu_j \big(z(t)\big) \left\{\frac{H_{ij}+H_{ji}}{2}\right\}\boldsymbol{x(t)} \quad (6)$$

Now consider the quadratic Lyapunov function:

$$V\big(\boldsymbol{x(t)}\big) = \boldsymbol{x(t)^T} P \boldsymbol{x(t)} \quad (7)$$

$P = P^T \in R^{nxn}$ is symmetric positive definite matrix

The close loop fuzzy control system (3) is to be asymptotically stable

If $\dot{V}\big(\boldsymbol{x(t)}\big) < 0 \quad \forall\, x(t) \neq 0$

Now take the time derivative of equation (7) both side, we get

$$\dot{V}\big(\boldsymbol{x(t)}\big) = \boldsymbol{\dot{x}(t)^T} P \boldsymbol{x(t)} + \boldsymbol{x(t)^T} P \boldsymbol{\dot{x}(t)} \quad (8)$$

From (6) & (8) we get

$$\dot{V}\big(\boldsymbol{x(t)}\big) = \Big[\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{j=1}^{r} \mu_j \big(z(t)\big) H_{ii}\boldsymbol{x(t)} + 2\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{i<j} \mu_j \big(z(t)\big) \left\{\frac{H_{ij}+H_{ji}}{2}\right\}\boldsymbol{x(t)}\Big]^T P\boldsymbol{x(t)} + \boldsymbol{x(t)^T} P \Big[\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{j=1}^{r} \mu_j \big(z(t)\big) H_{ii}\boldsymbol{x(t)} + 2\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{i<j} \mu_j \big(z(t)\big) \left\{\frac{H_{ij}+H_{ji}}{2}\right\}\boldsymbol{x(t)}\Big] \boldsymbol{x(t)}\}]$$

$$\dot{V}\big(\boldsymbol{x(t)}\big) = \Big[\Big(\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{j=1}^{r} \mu_j \big(z(t)\big) H_{ii} + 2\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{i<j} \mu_j \big(z(t)\big) \left\{\frac{H_{ij}+H_{ji}}{2}\right\}\Big)\boldsymbol{x(t)}\Big]^T P\boldsymbol{x(t)} + \boldsymbol{x(t)^T} P \Big[\Big(\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{j=1}^{r} \mu_j \big(z(t)\big) H_{ii} + 2\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{i<j} \mu_j \big(z(t)\big) \left\{\frac{H_{ij}+H_{ji}}{2}\right\}\Big)\boldsymbol{x(t)}\Big]$$

$$\dot{V}\big(\boldsymbol{x(t)}\big) = \boldsymbol{x(t)^T}\Big[\Big(\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{j=1}^{r} \mu_j \big(z(t)\big) H_{ii} P + 2\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{i<j} \mu_j \big(z(t)\big) \left\{\frac{H_{ij}+H_{ji}}{2}\right\}\Big) P\Big]^T \boldsymbol{x(t)} + \boldsymbol{x(t)^T}\Big[\Big(\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{j=1}^{r} \mu_j \big(z(t)\big) H_{ii} P + 2\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{i<j} \mu_j \big(z(t)\big) \left\{\frac{H_{ij}+H_{ji}}{2}\right\} P\Big)\boldsymbol{x(t)}\Big]$$

$$\dot{V}\big(\boldsymbol{x(t)}\big) = \boldsymbol{x(t)^T}\Big(\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{j=1}^{r} \mu_j \big(z(t)\big) H_{ii}{}^T P^T + 2\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{i<j} \mu_j \big(z(t)\big) \left\{\frac{H_{ij}+H_{ji}}{2}\right\}^T P^T\Big)\boldsymbol{x(t)} + \boldsymbol{x(t)^T}\Big[\Big(\sum_{i=1}^{r} \mu_i \big(z(t)\big) \sum_{j=1}^{r} \mu_j \big(z(t)\big) H_{ii} P +$$

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$2\sum_{i=1}^{r}\mu_i\left(z(t)\right)\sum_{i<j}\mu_j\left(z(t)\right)\left\{\frac{H_{ij}+H_{ji}}{2}\right\}P\Big)\boldsymbol{x}(\boldsymbol{t})\Big]$

$\dot{V}\left(\boldsymbol{x}(\boldsymbol{t})\right)=\boldsymbol{x}(\boldsymbol{t})^{\boldsymbol{T}}\Big(\sum_{i=1}^{r}\mu_i\left(z(t)\right)\sum_{j=1}^{r}\mu_j\left(z(t)\right)\left(H_{ii}{}^{T}P+\right.$

$\left.H_{ii}P\right)+2\sum_{i=1}^{r}\mu_i\left(z(t)\right)\sum_{i<j}\mu_j\left(z(t)\right)\left(\left\{\frac{H_{ij}+H_{ji}}{2}\right\}^{T}P^{T}+\right.$

$\left.\left\{\frac{H_{ij}+H_{ji}}{2}\right\}P\right)\Big)\boldsymbol{x}(\boldsymbol{t})$

Now for the close loop fuzzy control system (3) is to be asymptotically stable if $\dot{V}\left(\boldsymbol{x}(\boldsymbol{t})\right)<0$ then we have

$\boldsymbol{H_{ii}}^{\boldsymbol{T}}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{H_{ii}}<0$ & $\left\{\frac{\boldsymbol{H_{ij}}+\boldsymbol{H_{ji}}}{\boldsymbol{2}}\right\}^{\boldsymbol{T}}\boldsymbol{P}+\boldsymbol{P}\left\{\frac{\boldsymbol{H_{ij}}+\boldsymbol{H_{ji}}}{\boldsymbol{2}}\right\}<0.$

Hence, $\{\boldsymbol{A_i}-\boldsymbol{B_i}\boldsymbol{F_i}\}^{\boldsymbol{T}}\boldsymbol{P}+\boldsymbol{P}\{\boldsymbol{A_i}-\boldsymbol{B_i}\boldsymbol{F_i}\}<0$ and

$\boldsymbol{H_{ij}}^{\boldsymbol{T}}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{H_{ij}}<0$

4 NUMERICAL EXAMPLE

Example 1 consider the problem of balancing and swing-up of an inverted pendulum on a cart. Recall the equation of motion for the pendulum [4]:

$$\dot{x}_1(t)=x_2(t) \qquad (9)$$

$$\dot{x}_2(t)=\frac{g\sin(x_1(t))-\frac{amlx_2^2(t)\sin(2x_1(t))}{2}-a\cos(x_1(t))u(t)}{\frac{4l}{3}-amlcos^2(x_1(t))} \qquad (10)$$

where $x_1(t)$ denotes the angle of the pendulum from the vertical (in radian) and $x_2(t)$is the angular velocity; g = 9:8 m/s² is the gravity constant, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, and u is the force applied to the cart (in newtons); $a=1/(m+M)$.

Let $\boldsymbol{m}=\boldsymbol{2.0}$ kg, $\boldsymbol{M}=\boldsymbol{8.0}$ kg, $\boldsymbol{2l}=\boldsymbol{1.0}$ metre in the simulation

4.1 Two-Rule modeling and control

The control objective of this subsection is to balance the inverted pendulum for the approximate range $x_1\epsilon(-90^0,90^0)$. In order to use the PDC approach, we must have a fuzzy model which represents the dynamics of the nonlinear plant. Therefore we first represent the system (9) & (10) by a Takagi-Sugeno fuzzy model. To minimize the design effort and complexity, we try to use two-rule fuzzy model.

Model rule 1: IF $x_1(t)$ is about 0.

THEN $\dot{\boldsymbol{x}}(\boldsymbol{t})=A_1\boldsymbol{x}(\boldsymbol{t})+B_1\boldsymbol{u}(\boldsymbol{t})$

Control Rule 1: IF $x_1(t)$ is about0. THEN $\boldsymbol{u}(t)=-\boldsymbol{F_1}\boldsymbol{x}(t)$

Model rule 2: IF$x_1(t)$ is about $\pm\frac{\Pi}{2}$

THEN $\dot{\boldsymbol{x}}(\boldsymbol{t})=A_2\boldsymbol{x}(\boldsymbol{t})+B_2\boldsymbol{u}(\boldsymbol{t})$

Control Rule 2: IF $x_1(t)$ is about$\pm\frac{\Pi}{2}$. THEN $\boldsymbol{u}(t)=-\boldsymbol{F_2}\boldsymbol{x}(t)$

where $\boldsymbol{x}(\boldsymbol{t})=[x_1(t)\quad x_2(t)]^T$

$$A_1=\begin{bmatrix}0 & 1\\ \frac{g}{\frac{4l}{3}-aml} & 0\end{bmatrix};\qquad B_1=\begin{bmatrix}0\\ -\frac{a}{\frac{4l}{3}-aml}\end{bmatrix}$$

$$A_2=\begin{bmatrix}0 & 1\\ \frac{2g}{\Pi(\frac{4l}{3}-aml\beta^2)} & 0\end{bmatrix};\quad B_2=\begin{bmatrix}0\\ -\frac{a\beta}{\frac{4l}{3}-aml\beta^2}\end{bmatrix};\ \beta=\cos(88^0)$$

The output of Fuzzy Controller is given by:

$$\boldsymbol{u}(t)=-\sum_{i=1}^{2}\mu_i\left(z(t)\right)F_i\boldsymbol{x}(t) \qquad (11)$$

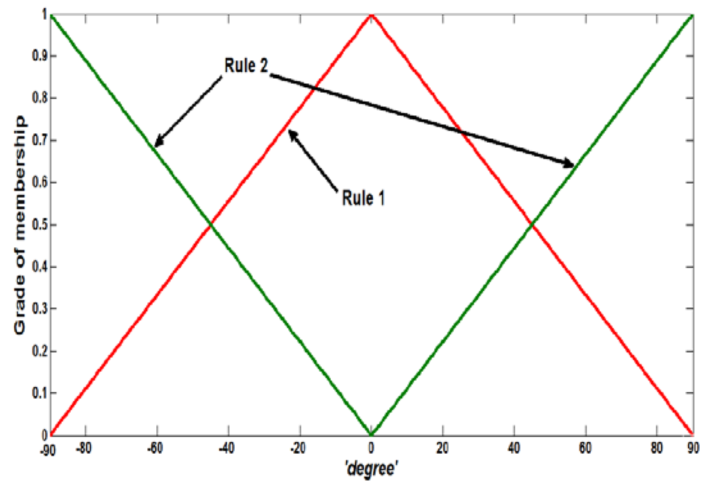

Fig. 1 Membership functions of two-rule model of example 1

4.2 Result

Choose the closed-loop Eigen values as$[-2,-2]$.

The state feedback gain matrices are obtained as:

$\boldsymbol{F_1}=[-\boldsymbol{425.8780},-\boldsymbol{80.00}]$ & $\boldsymbol{F_2}=[-\boldsymbol{300.1427},-\boldsymbol{80.00}]$

Now $\boldsymbol{A_1}-\boldsymbol{B_1}\boldsymbol{F_1}=\boldsymbol{A_2}-\boldsymbol{B_2}\boldsymbol{F_2}=\boldsymbol{H}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{1}\\ -\boldsymbol{4} & -\boldsymbol{4}\end{bmatrix}$

$$\boldsymbol{H_{12}}=\frac{\{\boldsymbol{A_1}-\boldsymbol{B_1}\boldsymbol{F_2}\}+\{\boldsymbol{A_2}-\boldsymbol{B_2}\boldsymbol{F_1}\}}{\boldsymbol{2}}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{1}\\ -\boldsymbol{53.2517} & -\boldsymbol{19.812}\end{bmatrix}$$

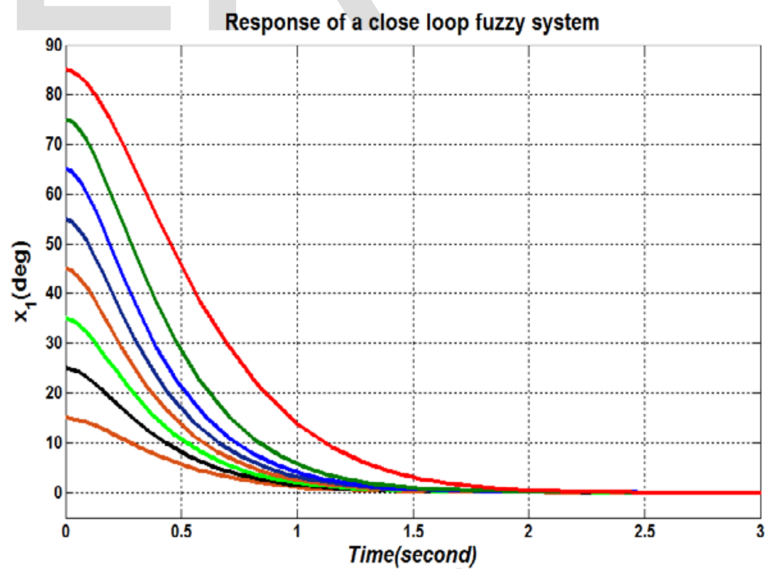

Fig. 2 shows the angle response of the fuzzy control system of example 1 with initial condition $\boldsymbol{x_1}=\boldsymbol{15^0},\boldsymbol{25^0},\boldsymbol{35^0},\boldsymbol{45^0},\boldsymbol{55^0},\boldsymbol{65^0},\boldsymbol{75^0},\boldsymbol{85^0}$ ***and*** $\boldsymbol{x_2}=\boldsymbol{0}$

The common Matrix P is obtained as:

$$\boldsymbol{P}=\begin{bmatrix}\boldsymbol{0.0889} & \boldsymbol{0.0068}\\ \boldsymbol{0.0068} & \boldsymbol{0.0061}\end{bmatrix}$$

and it can show that the stability condition of theorem 1 are satisfied:

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$$\{A_i - B_iF_i\}^T P + P\{A_i - B_iF_i\} < 0\ ;\ for\ i = 1,2$$
$$H_{12}{}^T P + PH_{12} < 0$$

5 CONCLUSION

The nonlinear dynamic system has been controlled by the fuzzy control system and the condition of stability of the closed loop fuzzy control system has been derived. The gain of the fuzzy state feedback controller is being obtained by the method of pole placement. A numerical example has been given to analyze the stability of given fuzzy control system.

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