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A Study on Second-Order Fuzzy Differential Equations using STHWS Method

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Abstract— In this paper, we study the numerical method for solving second-order fuzzy differential equations using single term Haar wavelet series (STHWS). And, we present two examples with initial condition [16] having two different solutions to illustrate the efficiency of the proposed method under STHWS. Error graphs are presented to highlight the efficiency of the STHWS.

Index Terms— Fuzzy differential equations, Fuzzy Improved Runge-Kutta Nystrom method,Fuzzy initial value problems, Haar wavelets, Runge-Kutta method, Single term Haar wavelet series.

1 INTRODUCTION

The study of fuzzy differential equation (FDE) forms a suitable setting for mathematical modeling of real world problems in which uncertainties or vagueness pervade. The concept of a fuzzy derivative was defined by Chang and Zadeh in [7]. It was followed up by Dubois and Prade in [8], who used the extension principle. The term "fuzzy differential equation" was introduced in 1987 by Kandel and Byatt [13, 14]. There have been many suggestions for the definition of fuzzy derivative to study FDE. The first and the most popular approach is using the Hukuhara differentiability for fuzzy-value functions. Under this setting, mainly the existence and uniqueness of the solution of a FDE are studied [1-3]. This approach has a drawback: the solution becomes fuzzier as time goes by. Hence, the fuzzy solution behaves quite differently from the crisp solution [10-12] and [15]. Seikkala [17] introduced the notion of fuzzy derivative as an extension of the Hukuhara derivative and fuzzy integral, which was the same as what Dubois and Prade [8] proposed. Buckley and Feuring [4-6]gave a very general formulation of fuzzy first-order initial value problem.They firstly find the crisp solution, fuzzify it and then check to see if it satisfies the FDE. To alleviate the situation, Hllermeier [9] interpreted FDE as a family of differential inclusions. The main shortcoming of using differential inclusions is that we do not have a derivative of a fuzzy-valued function.The single term Haar wavelet series method for FDE was introduced in [18].

In [18] a generalized concept of STHWS method for fuzzy-valued functions is presented to solve nth-order fuzzy differential equations.This concept allows us to resolve the above mentioned shortcoming. Hence, we use this STHWS concept in the present paper to study a second-order FDE with initial conditions. The organized paper is as follows: In Section 2, we give some basic results on fuzzy numbers and define a fuzzy derivative and a fuzzy integral then the fuzzy initial values is treated in Section 3 using the extension principle of Zadeh and the concept of fuzzy derivative. It is shown that the fuzzy initial value problem has a unique fuzzy solution when f satisfies Lipschitz condition which guarantees a unique solution to the deterministic initial value problem. In Section 4, the STHWS method for solving Nth – order fuzzy differential equations is introduced. In Section 5 the proposed method is illustrated by solving several numerical examples [16], and the conclusion is drawn in Setion 6.

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2 PRELIMINARIES

An arbitrary fuzzy number is represented by an ordered pair of functions $\left(\underline{u}(r), \overline{u}(r)\right)$ for all $r \in [0, 1]$, which satisfy the following requirements [12]:

(i) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0, 1],

(ii) $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0, 1],

(iii) $\underline{u}(r) \le \overline{u}(r) \quad \forall r \in [0, 1]$,

let E be the set of all upper semi-continuous normal convex fuzzy numbers with bounded α-level intervals.

Lemma 2.1 Let $\left[\underline{v}(\alpha), \overline{v}(\alpha)\right]$, $\alpha \in (0, 1]$ be a given family of non-empty intervals. If

(*i*) $\left[\underline{v}(\alpha), \overline{v}(\alpha)\right] \supset \left[\underline{v}(\beta), \overline{v}(\beta)\right]$ for $0 < \alpha \le \beta$,

and

(*ii*) $\left[\lim_{k\to\infty} \underline{v}(\alpha_k), \lim_{k\to\infty} \overline{v}(\alpha_k)\right] = \left[\underline{v}(\alpha), \overline{v}(\alpha)\right]$,

whenever (α_k) is a non-decreasing sequence converging to $\alpha \in$ (0, 1], then the family $\left[\underline{v}(\alpha), \overline{v}(\alpha)\right]$, $\alpha \in (0, 1]$, represent the α-level sets of a fuzzy number v *in E*. Conversely if $\left[\underline{v}(\alpha), \overline{v}(\alpha)\right]$, $\alpha \in (0, 1]$, are α-level sets of a fuzzy number $v \in E$, then the conditions (i) and (ii) hold true.

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Definition 2.2 Let *I* be a real interval. A mapping $v : I \rightarrow E$ is called a fuzzy process and we denoted the α-level set by $[v(t)]_\alpha = [\underline{v}(t,\alpha), \overline{v}(t,\alpha)]$. The Seikkala derivative $v'(t)$ *of v* is defined by $[v'(t)]_\alpha = \left[\underline{v}'(t,\alpha), \overline{v}'(t,\alpha)\right]$, provided that is a equation defines a fuzzy number $v'(t) \in E$.

Definition 2.3 Suppose *u* and *v* are fuzzy sets in *E*. Then their Hausdroff $D : E \times E \rightarrow R_+ \cup \{0\}$,

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\left\{\left|\underline{u}(\alpha) - \underline{v}(\alpha)\right|, \left|\overline{u}(\alpha) - \overline{v}(\alpha)\right|\right\},$$

i.e.,$D(u, v)$ is maximal distance between α-level sets of *u* and *v*.

3 Second-Order Fuzzy Differential Equations

In this section, we study the fuzzy initial value problem for a second-order linear fuzzy differential equation.

$$\left.\begin{aligned} & x''(t) + a(t)x'(t) + b(t)x(t) = \omega(t), \\ & x(0) = c_1, \\ & x'(0) = c_2, \end{aligned}\right\} \quad (1)$$

where $c_1, c_2 \in R_f, a(t), b(t), \omega(t) \in R$. In this paper, we suppose $a(t), b(t) > 0$. Our strategy of solving (1) is based on the selection of derivative type in the fuzzy differential equation. We first give the following definition for the solutions of (1).

Definition 3.1 let $x : [a,b] \rightarrow R_f$ be fuzzy-valued function and n,m = 1,2. One says x is an (n,m)-solution for problem (1). If $D_n^{(1)}x(t)$, $D_{n,m}^{(2)}x(t)$ exist and $D_{n,m}^{(2)}x(t) + a(t)D_n^{(1)}x(t) + b(t)x(t) = \omega(t), x(0) = c_1, D_n^{(1)}x(0) = c_2$.

4 Single-Term Haar Wavelet Series Method

The orthogonal set of Haar wavelets $h_i(t)$ is a group of square waves with magnitude of ± 1 in some intervals and zeros elsewhere [12]. In general,

$$\left.\begin{aligned} & h_n(t) = h_1(2^j t - k), n = 2^j + k, \\ & j \geq 0, 0 \leq k < 2^j, n, j, k \in Z \end{aligned}\right\}$$

$$h_1(t) = \begin{cases} 1, 0 \leq t < \dfrac{1}{2} \\ -1, \dfrac{1}{2} \leq t < 1 \end{cases},$$

Namely, each Haar wavelet contains one and just one square wave, and is zero elsewhere. Just these zeros make Haar wavelets to be local and very useful in solving stiff systems. Any function *y(t)*, which is square integrable in the interval [0,1). Can be expanded in a Haar series with an infinite number of terms

$$\left.\begin{aligned} & y(t) = \sum_{i=0}^{\infty} c_i h_i(t), i = 2^j + k, \\ & j \geq 0, 0 \leq k < 2^j, n, j, t \in [0,1] \end{aligned}\right\} \quad (2)$$

where the Haar coefficients

$$c_i = 2^j \int_0^1 y(t) h_i(t) dt$$

are determined such that the following integral square error ε is minimized:

$$\left.\begin{aligned} & \varepsilon = \int_0^1 \left[y(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right]^2 dt, \\ & m = 2^j, j \in \{0\} \cup N \end{aligned}\right\}$$

usually, the series expansion Equation (2) contains an infinite number of terms for a smooth *y(t)*. If *y(t)* is a piecewise constant or may be approximated as a piecewise constant, then the sum in Eq. (2) will be terminated after *m* terms, that is

$$\begin{aligned} & y(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) = c_{(m)}^T h_{(m)}(t), t \in [0,1] \\ & c_{(m)}(t) = [c_0 c_1 \ldots c_{m-1}]^T, \\ & h_{(m)}(t) = [h_0(t) h_1(t) \ldots h_{m-1}(t)]^T, \end{aligned} \quad (3)$$

where "T" indicates transposition, the subscript *m* in the parantheses denotes their dimensions. The integration of Haar wavelets can be expandable into Haar series with Haar coefficient matrix *P[3]*.

$$\int h_{(m)}(\tau) d\tau \approx P_{(m \times m)} h_{(m)}(t), t \in [0,1]$$

where the m-square matrix *P* is called the operational matrix of integration and single-term $P_{(1\times1)} = \frac{1}{2}$. Let us define [12]

$$h_{(m)}(t) h_{(m)}^T(t) \approx M_{(m \times m)}(t),$$

and $M_{(1\times1)}(t) = h_0(t)$. Equation (3) satisfies

$$M_{(m \times m)}(t) c_{(m)} = C_{(m \times m)} h_{(m)}(t),$$

where $c_{(m)}$ is defined in Equation (3) and $C_{(1\times1)} = c_0$.

5 Numerical Examples

In this section, we solved the fuzzy initial value problems to show the efficiency and accuracy of the proposed methods.

Example 5.1 Cosider the following second-order fuzzy differential equation with fuzzy initial value [2] and [16]

$$\left.\begin{aligned} & y''(t) = -y(t), \quad (t \geq 0) \\ & y(0) = 0 \\ & y'(0) = [0.9 + 0.1r, 1.1 - 0.1r] \end{aligned}\right\}$$

The exact solution is as follows:

$$Y(t,r) = [(0.9 + 0.1r)\sin(t), (1.1 - 0.1r)\sin(t)]$$

Example 5.2 Cosider the following second-order fuzzy differential equation with fuzzy initial value [2] and [16]

$$\left.\begin{aligned} & y''(t) = -y(t) + t, \quad (t \geq 0) \\ & y(0) = [0.9 + 0.1r, 1.1 - 0.1r] \\ & y'(0) = [1.8 + 0.2r, 2.2 - 0.2r] \end{aligned}\right\}$$

The exact solution under (1)-differentiability: $Y(t) = [Y_1(t,r), Y_2(t,r)]$ where

$$Y_1(t,r) = \left(\frac{4}{5} + \frac{1}{5}r\right)\sin(t) + \left(\frac{9}{10} + \frac{1}{10}r\right)\cos(t) + t$$

$$Y_2(t,r) = \left(\frac{6}{5} - \frac{1}{5}r\right)\sin(t) + \left(\frac{11}{10} - \frac{1}{10}r\right)\cos(t) + t$$

TABLE 1
ERROR CALCULATION OF EXAMPLE 5.1 FOR $y_1(t,r)$

r	Example 5.1 : $y_1(t,r)$	
	FIRKN5 (s = 4)	STHWS
0	2.18E-10	1.01E-12
0.1	2.20E-10	1.02E-12
0.2	2.23E-10	1.03E-12
0.3	2.25E-10	1.04E-12
0.4	2.27E-10	1.05E-12
0.5	2.30E-10	1.06E-12
0.6	2.32E-10	1.07E-12
0.7	2.35E-10	1.08E-12
0.8	2.37E-10	1.09E-12
0.9	2.40E-10	1.10E-12
1	2.42E-10	1.11E-12

TABLE 2
ERROR CALCULATION OF EXAMPLE 5.1 FOR $y_2(t,r)$

r	Example 5.1 : $y_2(t,r)$	
	FIRKN5 (s = 4)	STHWS
0	2.66E-10	1.11E-12
0.1	2.64E-10	1.10E-12
0.2	2.61E-10	1.09E-12
0.3	2.59E-10	1.08E-12
0.4	2.57E-10	1.07E-12
0.5	2.54E-10	1.06E-12
0.6	2.52E-10	1.05E-12
0.7	2.49E-10	1.04E-12
0.8	2.47E-10	1.03E-12
0.9	2.44E-10	1.02E-12
1	2.42E-10	1.01E-12

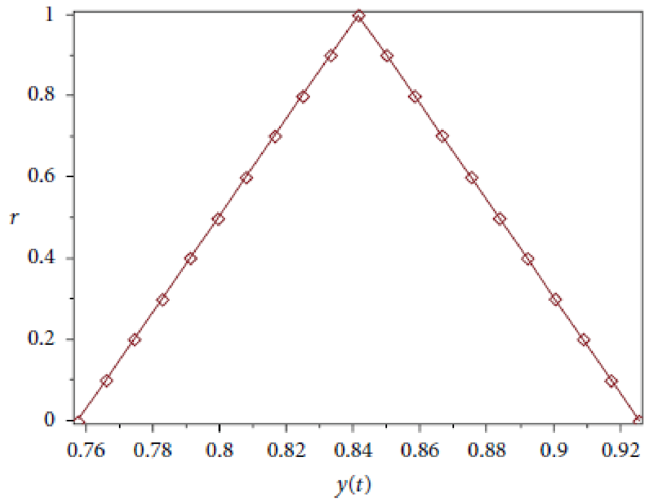

Fig. 1. Solution graph for Example 5.1

TABLE 3
ERROR CALCULATION OF EXAMPLE 5.2 FOR $y_1(t,r)$

r	Example 5.2 : $y_1(t,r)$	
	FIRKN5 (s = 4)	STHWS
0	3.13E-8	2.18E-10
0.1	3.19E-8	2.19E-10
0.2	3.25E-8	2.20E-10
0.3	3.31E-8	2.21E-10
0.4	3.37E-8	2.22E-10
0.5	3.43E-8	2.23E-10
0.6	3.49E-8	2.24E-10
0.7	3.55E-8	2.25E-10
0.8	3.61E-8	2.26E-10
0.9	3.67E-8	2.27E-10
1	3.73E-8	2.28E-10

TABLE 4
ERROR CALCULATION OF EXAMPLE 5.2 FOR $y_2(t,r)$

r	Example 5.2 : $y_2(t,r)$	
	FIRKN5 (s = 4)	STHWS
0	4.32E-8	3.06E-10
0.1	4.26E-8	3.05E-10
0.2	4.20E-8	3.04E-10
0.3	4.15E-8	3.03E-10

0.4	4.09E-8	3.02E-10
0.5	4.03E-8	3.01E-10
0.6	3.97E-8	2.99E-10
0.7	3.91E-8	2.98E-10
0.8	3.85E-8	2.97E-10
0.9	3.79E-8	2.96E-10
1	3.73E-8	2.95E-10

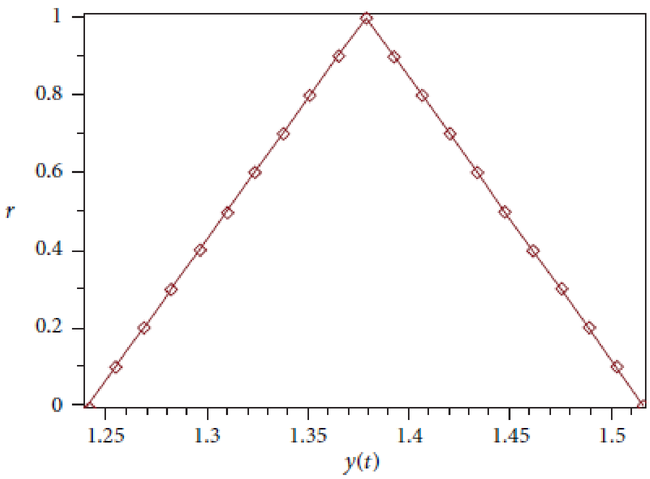

Fig. 2. Solution graph for Example 5.2

6 Conclusion

In this paper we introduce a new numerical method for solving second- order linear differential equation having fuzzy initial conditions. The efficiency and the accuracy of the STHWS method have been illustraded by suitable examples. The solutions obtained are compared well with the exact solutions and FIRKN5 [16]. It has been observed that the solutions by our method show good agreement with the exact solutions. From the numerical examples, we could conclude that the proposed method almost coincides with the exact solution and the FIRKN5 [16] method (refer Table 1 - 4 and Figure 1 - 2).

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