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A counterexample to a theorem of Tarun Pradhan

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Abstract— In a previous volume of *IJSER*, a theorem was published that claimed absence of a limit cycle for an exploited prey-predator fishery system of equations with Beddington-DeAngelis type functional response. A counterexample is offered to show that there is a limit cycle under conditions for which the theorem claims absence of limit cycles.

Key words: Beddington-DeAngelis functional response, bionomic equilibrium, biotechnical productivity, global stability, limit cycle, prey-predator fishery

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1 Introduction

An investigation of predator-prey dynamics in a fish population with Beddington-DeAngelis functional response was carried out in [3]. This analysis contained a theorem that we show to be incorrect. For clarity we mimic the notation used there. The system involved densities x and y of prey and predator densities, respectively, given by

$$\begin{cases} \frac{dx}{dt} = rx\left(1-\frac{x}{k}\right) - \frac{axy}{b+cx+y} - q_1Ex, \\ \frac{dy}{dt} = \frac{exy}{b+cx+y} - dy - q_2Ey. \end{cases} \tag{1}$$

The system of equations is studied over $P=\{(x,y)\,|\,x,y>0\}$ since, in any case, predation and harvesting naturally limits the growth of population densities. In the system, r and d are the natural growth and decline rates of prey and predator, k represents carrying capacity of the prey, and q_1E and q_2E are combined catchability and harvesting effort of prey and predator. In the system, the joint xy-terms represent the standard Beddington-DeAngelis functional response. When these are multiplied by rates a and e, what is obtained is the per capita interaction rate for feeding decline and feeding-related growth of prey and predator, respectively.

In [3], Theorem 3, the author stated the following, where BTP as defined in [1] is the biotechnical productivity, meaning the ratio of the biotic potential r to the catchability coefficient q_1.

Theorem 1 (Pradhan, [3]). *If the harvesting effort is less than or equal to the prey BTP ($E \leq r/q$), then the system (1) does not possess limit cycles in* $P=\{(x,y)\,|\,x,y>0\}$.

We note that the claim does hold trivially if the inequality is reversed since if $E > r/q_1$ the conditions of the Bendixon-Dulac test are satisfied, so that the system does not possess limit cycles in P. However, this is to be expected since $E > r/q_1$ suggests that harvesting of the prey exceeds its birth rate, so prey population density $x(t) \to 0$ as $t \to \infty$. It is then easily established that the predator population density $y(t) \to 0$ as well. Therefore, (0,0) is a stable steady-state solution. We conclude that under $E > r/q_1$, the system does not possess limit cycles in P.

However, under the original hypothesis , we can construct a counterexample to verify that, in fact, there is a limit cycle.

2 Constructing the Counterexample

Set $r_1 = r - Eq_1$, $d_1 = d + Eq_2$, and $k_1 = r_1k/r$. Then $E \leq r/q_1$ is equivalent to $r_1 \geq 0$. Under the change of constants, (1) becomes

$$\begin{cases} \frac{dx}{dt} = r_1x\left(1-\frac{x}{k_1}\right) - \frac{axy}{b+cx+y} \\ \frac{dy}{dt} = \frac{exy}{b+cx+y} - d_1y. \end{cases} \tag{2}$$

To nondimensionalize (2) we change variables from t to r_1t, x to x/k_1, and y to $y/(ck_1)$. We obtain

$$\begin{cases} \frac{dx}{dt} = x(1-x) - \frac{sxy}{A+x+y}, \\ \frac{dy}{dt} = \delta\left(\frac{xy}{A+x+y} - d_2y\right) \end{cases} \tag{3}$$

Where $s = a/r_1$, $\delta = e/(cr_1)$, $d_2 = cd_1/e$, and $A = a/(cK_1)$.

The following theorem will be used to lead to the desired counterexample.

Theorem 2 (Hwang, [2]). *If $d_2 < (1+A)^{-1}$ and $tr(J(x^*,y^*)) > 0$, then there is exactly one limit cycle for (3), where (x^*,y^*) is a steady state solution of (3) and where x^* and y^* satisfy*

$$(x^*)^2 + (s-1-d_2s)x^* - d_2As = 0, \quad y^* = (\tfrac{1}{d_2} - 1)x^* - A$$

and

$$\begin{aligned} tr(J(x^*,y^*)) &= -x^* + \frac{(s-\delta)x^*y^*}{(x^*+y^*+A)^2} \\ &= -x^* + \frac{(s-\delta)d_2y^*}{(x^*+y^*+A)} = -x^* + \frac{(s-\delta)d_2^2y^*}{x^*}. \end{aligned} \tag{4}$$

Matching the conditions in Theorem 2 for (3) can be accomplished by setting $s = 5/3$, $d_2 = 1/4$, $\delta = 1/2$, and $A = 1/10$.

Then we have $x^* = (-2+\sqrt{33})/24 \approx 0.1144$ and $y^* = (-19+5\sqrt{33})/40 \approx 0.2431$. From $s = a/r_1 = 5/3 > 0$, we have $r_1 > 0$, which implies that $E \leq r/q_1$.

Moreover, using these same values for S, d_2, δ, and A, as well as applying the values stated for x^* and y^* in (4) yields $tr(J(x^*,y^*)) = (309-47\sqrt{33})/960 \approx 0.0406 > 0$. Since $d_2 = 1/4 < (1+A)^{-1} = 10/11$, both conditions of Theorem 2 are satisfied. We conclude that there exists exactly one limit cycle.

3 Approximation of the Limit Cycle

We use the built-in numerical differential equation solver in *Mathematica* to approximate the solution to (3) with initial conditions $x(0) = y(0) = 0.5$ over the interval $0 \leq t \leq 200$ to visualize the limit cycle whose existence has been demonstrated. This is displayed in Fig. 1.

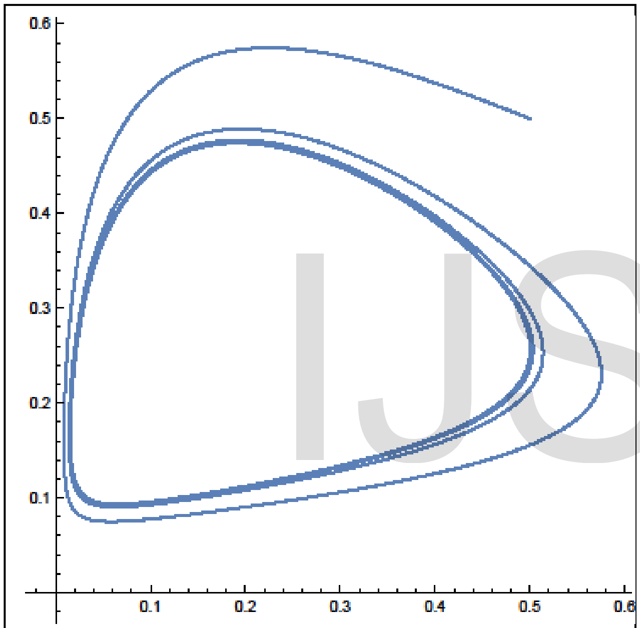

Fig. 1. A limit cycle over $0 \leq t \leq 200$ for (3) with $s = 5/3$, $d_2 = 1/4$, $\delta = 1/2$, and $A = 1/10$ using initial conditions $x(0) = y(0) = 0.5$.

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