A counterexample to a theorem of Tarun Pradhan

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Abstract—In a previous volume of IJSER, a theorem was published that claimed absence of a limit cycle for an exploited prey-predator fishery system of equations with Beddington-DeAngelis type functional response. A counterexample is offered to show that there is a limit cycle under conditions for which the theorem claims absence of limit cycles.

Key words: Beddington-DeAngelis functional response, bionomic equilibrium, biotechnical productivity, global stability, limit cycle, prey-predator fishery

1 INTRODUCTION

An investigation of predator-prey dynamics in a fish population with Beddington-DeAngelis functional response was carried out in [3]. This analysis contained a theorem that we show to be incorrect. For clarity we mimic the notation used there. The system involved densities \( x \) and \( y \) of prey and predator, respectively, given by

\[
\begin{align*}
\frac{dx}{dt} &= r x \left(1 - \frac{x}{k}\right) - \frac{axy}{b + cx + y} - q_1 Ey, \\
\frac{dy}{dt} &= \frac{axy}{b + cx + y} - dy - q_2 E y.
\end{align*}
\]

The system of equations is studied over \( P = \{(x,y) | x,y > 0 \} \) since, in any case, predation and harvesting naturally limits the growth of population densities. In the system, \( r \) and \( d \) are the natural growth and decline rates of prey and predator, \( k \) represents carrying capacity of the prey, and \( q_1 E \) and \( q_2 E \) are combined catchability and harvesting effort of prey and predator. In the system, the joint \( xy \)-terms represent the standard Beddington-DeAngelis functional response. When these are multiplied by rates \( a \) and \( c \), what is obtained is the per capita interaction rate for feeding decline and feeding-related growth of prey and predator, respectively.

In [3], Theorem 3, the author stated the following, where BTP as defined in [1] is the biotechnical productivity, meaning the ratio of the biotic potential \( r \) to the catchability coefficient \( q_1 \).

**Theorem 1** (Pradhan, [3]). If the harvesting effort is less than or equal to the prey BTP \((E \leq r/q_1)\), then the system (1) does not possess limit cycles in \( P = \{(x,y) | x,y > 0 \} \).

We note that the claim does hold trivially if the inequality is reversed since if \( E > r/q_1 \) the conditions of the Bendixon-Dulac test are satisfied, so that the system does not possess limit cycles in \( P \). However, this is to be expected since \( E > r/q_1 \) suggests that harvesting of the prey exceeds its birth rate, so prey population density \( x(t) \to 0 \) as \( t \to \infty \). It is then easily established that the predator population density \( y(t) \to 0 \) as well. Therefore, \((0,0)\) is a stable steady-state solution. We conclude that under \( E > r/q_1 \), the system does not possess limit cycles in \( P \).

However, under the original hypothesis, we can construct a counterexample to verify that, in fact, there is a limit cycle.

2 CONSTRUCTING THE COUNTEREXAMPLE

Set \( r_1 = r - Eq_1 \), \( d_1 = d + Eq_2 \), and \( k_1 = r_1 k_1/r \). Then \( E \leq r_1/q_1 \) is equivalent to \( r_1 \geq 0 \). Under the change of constants, (1) becomes

\[
\begin{align*}
\frac{dx}{dt} &= r_1 x \left(1 - \frac{x}{k_1}\right) - \frac{axy}{b + cx + y} - q_1 Ey, \\
\frac{dy}{dt} &= \frac{axy}{b + cx + y} - dy - q_2 E y.
\end{align*}
\]

To nondimensionalize (2) we change variables from \( t \) to \( r_1 t \), \( x \) to \( x/k_1 \), and \( y \) to \( y/(ck_1) \). We obtain

\[
\begin{align*}
\frac{dx}{dt} &= x(1 - \frac{x}{k}) - \frac{axy}{A + x + y}, \\
\frac{dy}{dt} &= \delta \left(\frac{xy}{A + x + y} - d_2 y\right).
\end{align*}
\]

Where \( s = a/r_1 \), \( \delta = c/(ck_1) \), \( d_2 = cK_1/e \), and \( A = a/(ck_1) \).

The following theorem will be used to lead to the desired counterexample.

**Theorem 2** (Hwang, [2]). If \( d_2 < (1 + A)^{-1} \) and \( tr(J(x^*,y^*)) > 0 \), then there is exactly one limit cycle for (3), where \((x^*,y^*)\) is a steady-state solution of (3) and where \( x^* \) and \( y^* \) satisfy

\[
(\delta^2 - 1)x^2 + (s - 1 - d_2 s)x^* - d_2 A s = 0, \quad y^* = \left(\frac{1}{d_2} - 1\right)x^* - A
d
\]

and

\[
tr(J(x^*, y^*)) = -x^* + \frac{(x^* - y^*)^2}{(x^* + y^* + A)} = -x^* + \frac{(x^* - y^*)^2}{(x^* + y^* + A)} = -x^* + \frac{(x^* - y^*)^2}{x^*}.
\]

Matching the conditions in Theorem 2 for (3) can be accomplished by setting \( s = 5/3 \), \( d_2 = 1/4 \), \( \delta = 1/2 \), and \( A = 1/10 \).
Then we have \(x^* = \frac{(-2+\sqrt{33})}{24} \approx 0.1144\) and \(y^* = \frac{(-19+5\sqrt{33})}{40} \approx 0.2431\). From \(s = a/r_1 = 5/3 > 0\), we have \(r_1 > 0\), which implies that \(E \leq r/q_1\).

Moreover, using these same values for \(S, d_2, \delta,\) and \(A\), as well as applying the values stated for \(x^*\) and \(y^*\) in (4) yields \(tr(J(x^*,y^*)) = \frac{(309-47\sqrt{33})}{960} \approx 0.0406 > 0\). Since \(d_2 = 1/4 < (1+A)^2 = 10/11\), both conditions of Theorem 2 are satisfied. We conclude that there exists exactly one limit cycle.

### 3 APPROXIMATION OF THE LIMIT CYCLE

We use the built-in numerical differential equation solver in *Mathematica* to approximate the solution to (3) with initial conditions \(x(0) = y(0) = 0.5\) over the interval \(0 \leq t \leq 200\) to visualize the limit cycle whose existence has been demonstrated. This is displayed in Fig. 1.

![Fig. 1. A limit cycle over 0 ≤ t ≤ 200 for (3) with s = 5/3, d_2 = 1/4, δ = 1/2, and A = 1/10 using initial conditions x(0) = y(0) = 0.5.](image)

**REFERENCES**

