Algebraic Structure of Boolean and Cubic Graphs

Faisal H Nesayef
Department of Mathematics, Faculty of Science, University of Kirkuk, Iraq

Abstract—The concept of applying Group Theory on Graphs has been developed by many researchers in order to demonstrate very basic and important graph properties and establish the connections between groups and graph theory. This was carried out by many researchers such as Biggs [1], who studied graph theory with Algebraic terms. This gave some principles ideas of applications of Algebra on Graph Theory.

Many researchers, such as [2], [8], studied the automorphisms of graphs and group theory. One of the areas of group theory which has been constantly used in graph theory is the symmetric groups and the action of the group elements on the edges and the vertices of a specific graphs.

Index Terms—Boolean graphs, Cubic graphs, Cyclic groups, Direct product of groups, Isomorphic graphs, Groups of automorphisms, n-dimensional cube, Permutations, Symmetric groups.

INTRODUCTION

This section deals with the basic principles of graphs and groups. It introduces the necessary definitions and properties needed in the second section. These definitions and properties are mainly extracted from other references such as [3], [4], and [7]. However, our attention has been restricted to the properties needed in our work in this paper. So we have backed these definitions by various examples to illustrate the main purpose.

The main results are given in Section Two. In this section we have established a connection between the Cubic graph and the symmetric group Sn including the cyclic groups and the direct products of groups.

1 BASIC PROPERTIES

Definition 1.1: A graph consists of two sets V(G) and E(G) called the set of vertices and the set of edges of G, respectively, together with two functions
\[ i: E \rightarrow V \quad \text{and} \quad t: E \rightarrow V. \]

We say that the edge "e" joins the vertex \( i( e ) \) to \( t( e ) \). The vertex \( i( e ) \) is called the initial vertex of "e" and \( t( e ) \) is called the terminal vertex of "e". For each \( e \) in E there is an element \( \bar{e} \neq e \) in E called the inverse of "e" such that
\[ i(\bar{e}) = t(e), t(\bar{e}) = i(e) \text{and} \bar{\bar{e}} = e. \]

We say that u and v are adjacent to each other if uv is an edge in E(G), moreover, \( u \sim v \) if \( uv \) is an edge in E(G).

A walk is a sequence of vertices and edges such that \( w = v_0e_1v_2e_2...v_ne_n \), where some edges and vertices can be visited more than once.

Definition 1.2: Let \( V = \{v_0, v_1, ..., v_k\} \) be a set of pairwise distinct points and let \( E = \{v_0v_1, v_1v_2, ..., v_{k-1}v_k\} \). Then \( P(V, E) \) is a path from \( v_0 \) to \( v_k \) of length k. So a path of length "k" has k edges and (k+1) vertices.

Definition 1.3: Let \( p = p(v_1, v_k) \) be as in the above definition, then \( C = C^{k+1} = p + v. v_k \) is a cycle of length (k+1). The length of a cycle is the number of vertices or edges on it.

Definition 1.4: A path \( v_0, e_1, n_2, e_2, ..., v_ne_n \) is closed if \( v_0 = v_n \). A cycle can be described as a circuit in which no vertex appears more than once except the first which is also the last.

\[ v_0e_1v_1e_2v_2e_3v_3e_4v_4e_5v_5e_6v_6 \]

Figure (1)

An n-cycle is a cycle with n vertices. A trail is a walk with all vertices distinct. A circuit is a trail from any vertex back to itself. This means a circuit is a closed walk with all edges distinct.

Definition 1.5: A graphs \( G(V, E) \) is called connected if for any \( v_1, v_2 \in V \), there exists a path for \( v_1 \) to \( v_2 \).

We say that u and v are adjacent to each other if uv is an edge in E(G), moreover \( u \sim v \) if uv is an edge.
Definition 1.6: Let $G = (V, E)$ be a graph. If $v \in V$ then $N_G(v) = \{u : uv \in E\}$ is the set of neighbours of $v$. $d(v) = |N_G(v)|$ is the degree of $v$.

Definition 1.7: If $d(v) = k$, when $k$ is an integer for all $v \in V$ then $G$ is called regular of degree $k$. So the graph is regular or k-regular, if all its vertices have the same degree $k$.

The following theorem is very well known and given in many other references.

Theorem (Handshake): Let $G = (V, E)$ be a graph.

Then $\sum_{v \in V} d(v) = 2|E|$


Proof Let $X = \{(v, e) : v \text{ is an end vertex of } e\}$

Count $X$ in two directions. When we start with $e$, we will get $|X| = 2|E|$. Starting with $v$ we get $|X| = \sum_{v \in V} d(v)$.

Definition 1.8: Let $G = (V, E)$ and $G' = (V', E')$ be graphs. Then a bijective map $\phi : V \to V'$ is an isomorphism provided that $\{u, v\} \in E$ iff $(\phi(u), \phi(v)) \in E'$. $\phi$ is called an Automorphism if $G = G'$.

Definition 1.9: Let $V = \{v_0, v_1, \ldots, v_k\}$ be a set of pairwise distinct points, and let $E = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$.

Then $P = (E, V)$ is a path from $v_0$ to $v_k$ of length $k$. So a path of length $k$ has $k$ edges and $(k + 1)$ vertices.

Definition 1.10: Let $P = P(v_0, v_k)$ be as in definition 1.6. Then $C = C^{k+1} = P + v_0v_k$ is a cycle of length $(k + 1)$.

The length of a cycle is the number of vertices or edges on it.

Definition 1.11: Let $G = (V, E)$ be a graph. If $v, v'$ are vertices, then the distance $d_G(v, v') = d(v, v')$ from $v$ to $v'$ is the least $k$ so that there is a path $P(v, v') = P^k \subseteq G$.

The diameter $\text{diam}(G)$ of $G$ is the largest $k$ such that $d(v, v') = k$ for some pair of vertices. The girth $g(G)$ of $G$ is the least $k$ such that $C^k \subseteq G$ for some cycle $C^k$.

2. SYMMETRIC GROUPS

Definition 2.1 A permutation of a set $S$ is a one-to-one, onto mapping from $S$ to itself.

Let $S = \{1, 2, 3\}$, the permutations of $S$ are: $e = (1)(2)(3)$, $a = (1)(23)$, $b = (2)(13)$, $c = (3)(12)$, $d = (123)$, $f = (132)$, where $a^2 = b^2 = c^2 = 1$ and $f^{-1} = (2, 3, 1) = d$.

A permutation $\pi \in S_n$ which interchanges two elements and fixed all the others is called a transposition.

Definition 2.2 A Permutation $\pi \in S_n$ is called cycle if it has at most one orbit containing more than one element. The length of a cycle is determined by the length of its largest orbit.

Eg: $\pi = (1, 2, 3)(4)$ is a cycle of length 2.

Let $\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 1 & 3 & 6 \end{bmatrix}$

Then $\pi$ is split into the following orbits:
{$\{1, 2, 4\}$, $\{3, 5\}$, $\{6\}$}

A permutation of a finite set is even or odd according to whether it can be expressed as a product of an even number of transpositions or the product of an odd number of transpositions.

Example
$(1, 4, 5, 6)(2, 1, 5) = (1, 6)(1, 5)(1, 4)(2, 5)(2, 1)$

So this is an odd permutation.

Definition 2.3 The direct product of the groups $G_1, \ldots, G_n$ when $n$ is finite is defined by:

$G_1 \times \ldots \times G_n$ consisting of elements of the form $(g_1, \ldots, g_n)$ where $g_i \in G_i$, $1 \leq i \leq n$.

Definition 2.4 A homomorphism from a group $G$ to a group $\hat{G}$ is a mapping $f : G \to \hat{G}$ where $f(g_1g_2) = f(g_1)f(g_2)$, where $g_1, g_2 \in G$.

Definition 2.5 An isomorphism is a one-to-one onto homomorphism.

Definition 2.6 An automorphism of a group $G$ is an isomorphism of a group $G$ onto itself.

The set of all automorphisms of a group $G$ is denoted by $Aut(G)$.

Theorem 2.1 Let $G$ be a group and $Aut(G)$ be the set of all automorphisms of $G$. Then $Aut(G)$ forms a group under the composition of functions.

Proof See [5].

Definition 2.7 Let $G = (V, E)$ be a finite graph. An automorphism of $G$ is a permutation of the vertex set that satisfies the condition $\{u_i, u_j\} \in E(G)$ if and only if $\{\phi(u_i), \phi(u_j)\} \in E(G)$.
The Automorphism Group of $G$ is the set of permutations of the vertex set that preserve adjacency. See [3].

$$\text{Aut}(G) = \{ \pi \in \text{Sym}(V) : \pi(E) = E \}.$$  

**Theorem 2.2** The set $\text{Aut}(G)$ of all group automorphisms of a group $G$ forms a group under compositions of functions.

**Proof** See [3].

### 3. APPLICATIONS

**Definition 3.1** An edge automorphism on a graph $G = (V, E)$ is a permutation $\pi$ on the set of edges of $G(E)$ satisfying $e_i, e_j$ are adjacent if and only if $\pi(e_i), \pi(e_j)$ are also adjacent.

**Theorem 3.1** $\text{Aut}(K_n) \cong S_n$ (isomorphic).

**Proof** $K_n$ contains $n$ vertices which are all connected to each other. Each vertex is connected to $n - 1$ edges. Each vertex from $K_n$ is mapping to another vertex. The other $n - 1$ vertices are connected to $n - 2$ vertices and so on.

Therefore, $\text{Aut}(K_n)$ contains $n(n - 1)(n - 2) ... 2 \times 1$ elements and this given by $n!$. Since $S_n$ contains $n!$ Elements, so each element in $\text{Aut}(K_n)$ is mapped to an element in $S_n$. Therefore, $\text{Aut}(K_n) = S_n$

The following is covered in [6].

**Cube Graph:** Let $V$ be the set of $v = (x_1, x_2, ..., x_n)$ with $x_i \in \{0,1\}$ and $n > 0$. $v_i$, $v_j \in V$, are adjacent if and only if they differ by only one coordinate when expressed as vectors. ie $(x_1, ..., x_n)$ is adjacent to $(x'_1, ..., x'_n)$ and only if they differ in precisely one position.

These will form the edges.

The Graph $G = C_n^u$, $n > 0$. (n-dimensional graph).

$V_0 = \{(0, x_2, ..., x_n) | x_i \in \{0,1\}\}$

$V_1 = \{(1, x_2, ..., x_n) | x_i \in \{0,1\}\}$

$V = V_0 \cup V_1$

$V_0 \cap V_1 = \emptyset$

**Figure (2)**

**Theorem 3.2** Every cube graph $C_n^u$ has a Hamiltonian cycle if $\geq 2$.

**Proof** See [6].

Let $v$ be the set of vertices of a graph. A permutation of $V$ is an automorphism of a graph $G$ if the following conditions hold: $uv \in E$ if and only if $(u)g(v) \in E$.

Now let $C_2^n = c_2 \times c_2 \times ... \times c_2$ be the direct product of $n$ copies of cyclic groups of order 2.

**Theorem 3.3** $C_n^u$ acts on $C_n^u$ as a group of automorphisms.

**Proof** See [6].

**The Group of Automorphisms of $C_n^u$**

$C_2$ can be regarded as a group of integers $\mathbb{Z}_2$

Therefore $C_n^u = c_2 \times c_2 \times ... \times c_2$ is the set of all sequences $g = (g_1, g_2, ... , g_n)$ with group operation $g + g' = (g_1, g_1', g_2, g_2', ..., g_n, g_n')$. Identify the vertices of $C_n^u$ with the elements of $C_n^u$.

If $g \in C_n^u$ where $x = (x_1, ..., x_n)$ is a vertex of $C_n^u$, then let $g = (g_1 + x_1, ..., g_n + x_n)$. This is a one-to-one automorphism map of the vertex set where $x$ and $x'$ differ in exactly one position if $g(x)$ and $g(x')$ differ in one position.

**Boolean Graphs:** Let $X = \{1, 2, ... , n\}$. Take $V$ as the collection of subsets of $X$. $v_1 \neq v_2 \in V$ are adjacent to each other if $(u \setminus v) \cup (v \setminus u)$ has only one element. This graph is called the Boolean graph $B^n$.

Now let $G = (V, E) = B^n$ be a Boolean graph. Let $V_0$ be the collection of all subsets of $\{1, 2, 3, ..., n\}$ and let $V_1$ be the collection of all subsets of $\{1, 2, 3, ..., n\}$. Therefore $V = V_0 \cup V_1$ is the set of vertices of $B^n$. If $u, v \in V_1$ we denote $u \equiv v$, if and only if the symmetric difference between $u$ and $v$ has size one. Then we get the graph $G_1 = (V_1, E_1)$ will be isomorphic to $B^{n-1}$.

There are edges between $x \in V_0$ and $y \in V_1$ if and only if $x = y \cup \{n\}$. So $G = B^n$ and $G[V_1]$ is isomorphic to $B^{n-1}$.

Now look at the number of vertices and edges in $B^n$.

From above we get that $|V| = |V_0| + |V_1|$ where $|V_0| = |V_1|$

By induction we get $|V| = 2^n$. So there are $2^n$ subsets of $\{1, 2, 3, ..., n\}$.

So for any set $u \in V$, there will be $n$ sets which have symmetric difference with $u$ of size 1.

Removing $i$ from $u$ if it is contained in $u$ gives an adjacent or adding $i$ to $u$ if it is not in $u$ gives a neighbor. Therefore the degree of each vertex in $G = B^n$ is $n$.
Using Handshake lemma we get $n|V| = 2|E| a\mathcal{C}_2^n$ and so $|E| = n 2^{n-1}$.

**Theorem 3.4** \(B^n\) is isomorphic to \(C_n^u\)

**Proof** Let \(\emptyset : V(C_n^u) \to V(B^n)\) be a map defined by:

\(x \sim y\) in \(C_n^u\) if and only if \(\emptyset(x) \sim \emptyset(y)\).

So let \(x = (x_1, \ldots, x_n)\) with \(x_i \in \{0, 1\}\) be a vertex of \(C_n^u\).

Define \(\emptyset(x)\) to be the set \(\{i : x_i = 1\}\). So \(\emptyset\) is onto.

If \(\emptyset(x) \sim \emptyset(y)\) then \(x = y\). Therefore \(\emptyset\) is a bijection between \(V(C_n^u)\) and \(V(B^n)\).

Now let \(x, y \in V(C_n^u)\), then \(x \sim y\) if and only if \(1 = |\{i : x_i \neq y_i\}|\) if and only if the symmetric difference of \(\emptyset(x)\) and \(\emptyset(y)\) has size one if and only if \(\emptyset(x) \sim \emptyset(y)\) in \(V(B^n)\).

Therefore \(\emptyset : V(C_n^u) \to V(B^n)\) is an isomorphism.

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**REFERENCES**


