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An Application of Estimating Process Capability Indices Based on Weibull Shape Parameter

Suboohi Safdar

Abstract— Process capability indices always been debated by quality practitioners for the measurements come from normal and non-normal controlled processes but indices based on parameter(s) of non-normal distribution is not formally discussed. This paper is an application of a procedure to estimate capability indices based on Weibull shape parameter of a process whose measurements come from two parameter Weibull distribution. A data consisting of Weibull measurements is taken, Sampling distribution based on Weibull shape parameter using rank regression method is obtained, assumptions before estimating indices are checked and four basic indices are summarized along with their confidence intervals at certain level of significance. The program is made in R-console and results shows that indices based on Weibull shape parameter can results equivalently as estimated indices based on process measurements of same process.

Index Terms— Basis Capability Indices, Bootstrapping, Control Charts, Non-Normal Process, Rank Regression, Statistical Process Control, Weibull Shape Parameter,

1 Introduction

PROCESS capability indices are quantitative and dimensionless measures which indicate the performance of a process in relation with process parameter(s) and process specification(s) see Kotz and Johnson [1]. Numerous work proposed and developed for measurements come from normal process see for details Juran [2], Kane [3], Chan et al. [4], Boyles [5], and Pearn et al. [6] and for non-normal process see Johnson [7], Gunter [8], Pyzdek [9], Boyles [10], Zwick [11], Farnum [12] among many others. Ahmed and Safdar [13], [14], [15] worked on estimating capability indices for non-normality under diverse distributional conditions. But work on estimating capability indices based on parameter(s) of non-normal processes is not reported formally. Ahmed and Safdar [16] worked on estimating capability indices based on Weibull shape parameter β for two data sets whose measurements come from Weibull processes. To estimate PCIs based on β, the mean and standard deviation of β are obtained using rank regression method.

$f(t)=\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]$ is the Weibull density function where α and β are scale and shape parameter respectively and $F(t)=1-\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]$ is cumulative distribution function. Taking $x_i=\ln\{-\ln[1-F_T(t_i)]\}$ and $y_i=\ln(t_i)$ the distribution function becomes regression equation y on x;

$$y_i=\ln(\alpha)+\frac{x_i}{\beta}+\varepsilon_i, i=1,2,..,n \quad (1)$$

Here α and β are the regression coeffecients for a simple linear regression equation y on x; such that $y_i=\beta_0+\beta_1 x_i+\varepsilon_i, i=1,2,..,n$. Equating both regression equations y on x, we get $\beta_0=\ln(\alpha)$ and $\beta_1=\frac{1}{\beta}$, and with inferential properties of Equation (1)

$$\hat{\beta}=\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})} \quad (2)$$

Using Delta method of Stuart and Ord [17];

$$V(\hat{\beta})=\left[\frac{\partial}{\partial\beta}\left(\frac{1}{\beta_1}\right)\right]^2 V(\beta_1)=\left[\left(-\frac{1}{\beta_1^2}\right)\right]^2 V(\beta_1) \quad (3)$$
$$=\left[\left(\frac{1}{\beta_1}\right)\right]^4 V(\beta_1)=\left[\left(\frac{1}{\beta_1}\right)\right]^4\left[\frac{1}{\sum(y_i-\bar{y})^2}\right]\sigma^2$$

It is known that the regression coefficients of simple linear regression equation are normally distributed so initially developed PCIs along wih their repective confidence intervals are summarized in section II to estimate PCIs based on Weibull shape parameter.

2 Process Capability Indices

For four basic PCIs $C_p, C_{pk}, C_{pm}, C_{pmk}$ a superstructure is presented by Vannman [18] as under;

$$C_p(u,v)=\frac{d-u|\mu-m|}{3\sqrt{\sigma^2+v(\mu-T)^2}}\ u,v\geq 0 \quad (3)$$

Where
$C_p(0,0)=C_p, C_p(1,0)=C_{pk}, C_p(0,1)=C_{pm} C_p(1,1)=C_{pmk}$

The $100(1-\alpha)\%$ Confidence interval for Cp

$$\left(\frac{\chi_{n-1,\alpha/2}}{(n-1)^{\frac{1}{2}}} \hat{C}_p , \frac{\chi_{n-1,1-\alpha/2}}{(n-1)^{\frac{1}{2}}} \hat{C}_p \right) \tag{4}$$

Where $\chi^2_{n-1,\alpha/2}$ and $\chi^2_{n-1,1-\alpha/2}$ are the upper $\alpha/2$ and $1-\alpha/2$ quantile of a chi square distribution with (n-1) degrees of freedom respectively.
For details see Kotz and Johnson [1].

The $100(1-\alpha)\%$ Confidence interval for Cpk

$$\hat{C}_{pk}\left[\frac{1 \pm z_{1-\alpha/2}}{\sqrt{2(n-1)}}\right] \tag{5}$$

For details see Nagata and Nagahata [19], [20].

The $100(1-\alpha)\%$ Confidence interval for Cpm

$$\left(\frac{\chi_{n,\alpha/2}}{\sqrt{n}} \tilde{C}_{pm} , \frac{\chi_{n,1-\alpha/2}}{\sqrt{n}} \tilde{C}_{pm} \right) \tag{6}$$

See Boyles [5], Subbaiah [21] and Patnaik [22].
An asymptotically unbiased interval for C_{pmk} is

$$\hat{C}_{pmk} \mp z_{\alpha/2} \frac{\hat{\sigma}_{pmk}}{\sqrt{n}} \tag{7}$$

Where

$$\hat{\sigma}^2_{pmk} = \left[\frac{1}{9(1+\delta^2)} + \frac{2\delta}{3(1+\delta^2)^{3/2}}\right]\hat{C}_{pmk} + \frac{72\delta^2 + D\left(\frac{m_4}{s_n^4} - 1\right)}{72(1+\delta^2)^2}\hat{C}^2_{pmk}$$

Here $\hat{\sigma}^2_{pmk}$ is the asymptotic estimator of $\mathrm{Var}(\hat{C}_{pmk})$.
$z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution, $m_4 = \sum_{i=1}^{n}(X_i - \overline{X})^4/n$, $\delta = (\overline{X} - T)/S_n$ and $S_n^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2/n$.

See for details Chen and Hsu [23].

• *Dr. Suboohi Safdar is currently working as an Assistant Professor in Department of Statistics, Universiy of Karachi. Pakistan. E-mail: suboohi@uok.edu.pk*

Section III illustrated the steps to obtain bootstrap estimates of PCIs for Weibull shape parameter.

3 Route to Estimate PCIs for Weibull Shape Parameter

For estimating PCIs for Weibull shape parameter a script is made in R-console R-3.01 [24] with packages *car*, *nls* and *boot* for estimating indices based on Weibull shape parameter.

i. Estimate Weibull scale and shape parameter for Albing data and draw a Weibull density curve.

ii. Simulate 10,000 samples 'ti' for Weibull distribution from estimated parameters. For each sample obtain y_i and x_i to regress equation 'y' on 'x' o estimate $\hat{\beta}$ and $v(\hat{\beta})$ as in section I.

iii. Boot the estimated Weibull shape parameter 100, 200, 500, 1000, 1500 and 2000 times and for each boot sample make subgroup of size 10 to construct $\overline{X} - R$ control charts to assess statistical controlled process assumption before estimating PCIs. (Exclude measurements if falls outside the preset specification limits and re-construct control charts.

iv. For each boot sample obtain point and 95% intervals estimates of four basic PCIs from equations (1) to (5) for Weibull shape parameter. We consider the null hypothesis $H_o : \hat{C}_\beta = 1$ with the alternative hypothesis $H_o : \hat{C}_\beta > 1$ at 5% level of significance.

Section IV illustrates an application of the procedure using Albing [25] data set.

4 Illustration

For illustration a data consisting of 52 measurements of deviation from surface of gearwheel experiment by Albing and Vannman [25] is considered.

TABLE 1
52 Measurements of Deviation from surface

0.004	0.008	0.011	0.014	0.017	0.019
0.017	0.019	0.023	0.027	0.038	0.005
0.04	0.005	0.009	0.013	0.014	0.018
0.015	0.019	0.021	0.025	0.034	0.008
0.022	0.027	0.036	0.005	0.008	0.011
0.009	0.013	0.014	0.018	0.02	0.024
0.021	0.024	0.033	0.042	0.006	0.01
0.01	0.014	0.017	0.019	0.022	0.027
0.014	0.032	0.013	0.035		

First a Weibull density curve of gearwheel experiment is drawn. Figure 1 displays the Weibull density curve for the Gearwheel experiment.

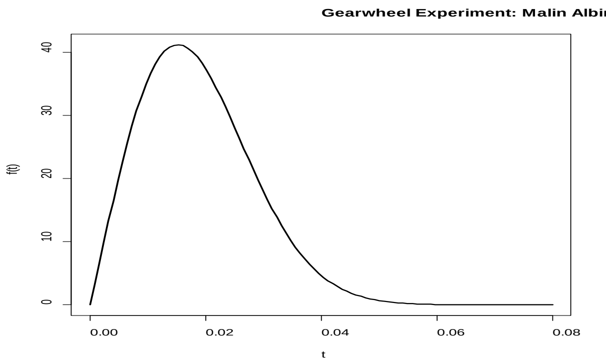
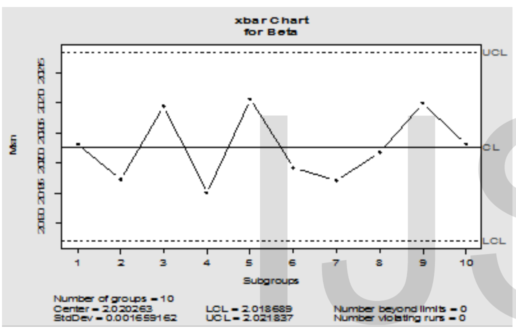
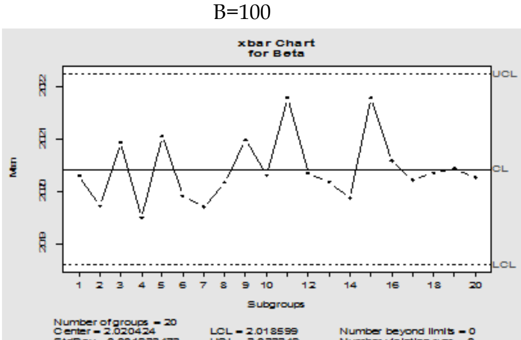


Fig. 1: Weibull Density Curve for Gearwheel Experiment

Control charts for each boot sample of Weibull shape parameter are constructed to monitor the foremost assumption of estimating capability indices based on Weibull shape parameter. See Figure 2



B=100



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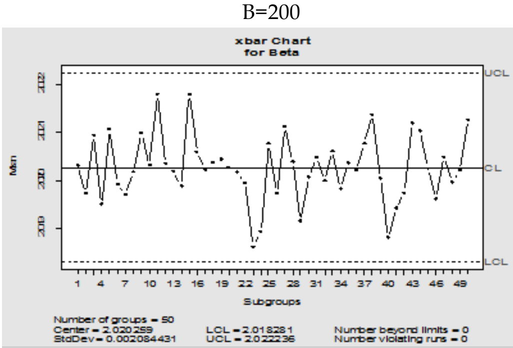
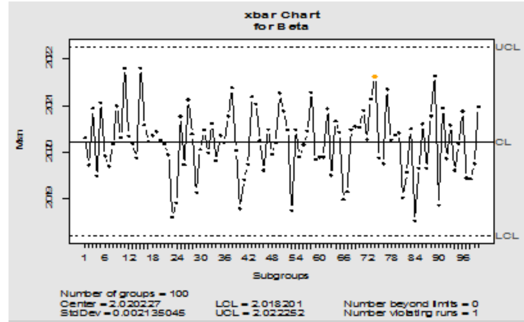
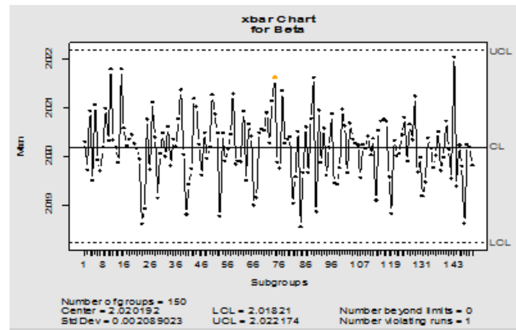


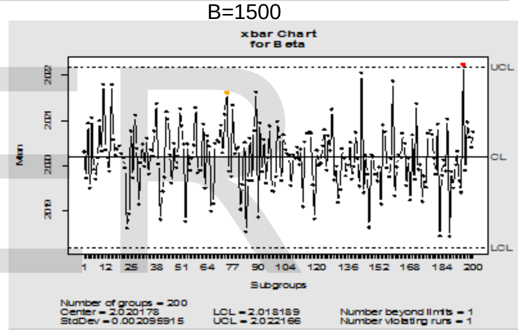
B=500



B=1000



B=1500



B=2000

Figure 2: $\overline{X}-R$ Control Chart for Boot Sample based on Weibull Shae Parameter

From the $\overline{X}-R$ control chart, the foremost assumption of estimating capability indices is checked that each boot sample is in statistical control, as the program in R-console is made that it exclude and replace those samples which are not in statistical control. For the statistical controlled samples of Weibull shape parameter basic process capability indices with their confidence intervals using Equations (3) to (7) are estimated.

Table 2 displays four basic process capability indices for B=100, 200, 500, 1000, 1500 and 2000.

TABLE 2
BASIC PCIS BASED ON WEIBULL SHAPE PARAMETER

B	C_p	C_{pk}	C_{pm}	C_{pmk}
100	1.196	1.004	1.036	0.870
200	1.059	0.916	0.974	0.843
500	0.968	0.817	0.882	0.744
1000	0.940	0.783	0.851	0.709
1500	0.967	0.800	0.865	0.716
2000	0.971	0.800	0.864	0.712

For each boot sample, four basic indices are summarized in Table 1 showing PCIs based on Weibull shape parmater and it may concluded that process based on Weibull shape parameter is considered capabale as per decision rule.

TABLE 3
95% CI OF PCIS BASED ON WEIBULL SHAPE PARAMETER

INDICES	B	100	200	500	1000	1500	2000
C_p	**LL**	1.18	1.04	0.95	0.93	0.95	0.96
	UL	1.21	1.07	0.98	0.95	0.98	0.98
C_{pk}	**LL**	0.99	0.90	0.80	0.77	0.79	0.79
	UL	1.02	0.93	0.83	0.80	0.81	0.81
C_{pm}	**LL**	1.02	0.96	0.87	0.84	0.85	0.85
	UL	1.05	0.99	0.89	0.86	0.88	0.88
C_{pmk}	**LL**	0.86	0.84	0.74	0.70	0.71	0.71
	UL	0.88	0.85	0.75	0.71	0.72	0.72

Table 3 gives 95% confidence intervals for basic PCIs of each boot sample based on Weibull shape parameter and it is noted that for each boot sample all four basic point estimates of basic indices lies within their respective confidence intervals.

5 DISCUSSION AND CONCLUSION

Vannman and Albing [25] worked on experiment consisting of measurements of deviation from the surface of Gearwheel and set a hypothesis at significance level that measurements comes from Weibull process are deemed capable. For the same data set we obtained four basic PCIs with 95% confidence interval based on bootstrap samples of Weibull shape parameter and concluded similar findings as Albing did.

6 APPLICATIONS OF WEIBULL SHAPE PARAMETER

Zhang, Xie and Tang [26] used Weibull shape parameter as a measure of reliability and compared various estimators based on different assumptions. Ahmed and Safdar [16] worked on estimating indices for annual flow minimum mean daily flows and Pearn rubber edge experiment based on Weibull shape parameter. Guo and Wang [27] constructed control charts for monitoring Weibull Shape Parameter Based on Type-II Censored Sample. Yavuz [28] worked on estimation of Shape Parameter of Weibull distribution using linear regression methods. Cui and Yang [29] worked on interval estimation of PCIs based on Weibull distributed quality data of supplier products.

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