Analyticity Theorem and Operation-Transform Formula for Laplace–Mellin Integral Transform to A Class of Generalized Function

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ABSTRACT : We have Extended Laplace-Mellin Integral Transformation (LMIT) to a class of Generalized Function. In this paper we discuss the 'Testing Function Space' $\mathfrak{LM}_{a,b,c,d}$ and its dual $\mathfrak{LM}'_{a,b,c,d}$. We have proved that $\mathfrak{LM}_{a,b,c,d}$ is 'Complete Space'. We have derived some lemmas those are '$e^{-sl}\,m^{p-1} \in \mathfrak{LM}_{a,b,c,d}$' , 'D($\mathfrak{I}$) is a subspace of $\mathfrak{LM}_{a,b,c,d}$' and '$\mathfrak{LM}_{a,b,c,d}$ is a dense subspace of E($\mathfrak{I}$)' . 'Analyticity Theorem' of Laplace-Mellin Integral Transformation has been derived and 'Some Operation transformation formula' for this transform has been proved.

Keywords : Laplace-Mellin Integral Transformation (LMIT), Laplace Transformation, Mellin Transformation, Analyticity Theorem and Operation Transform Formulae.

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1 INTRODUCTION

Let us consider a transform:

$$\mathfrak{LM}\,[f(l,m)] = F(s,p) = \int_0^\infty \int_0^\infty f(l,m)\, e^{-sl}\, m^{p-1}\, dl\, dm$$

Where $f(l,m)$ is a suitably restricted conventional function defined on the positive real line $0 < l < \infty$ & $0 < m < \infty$ and $0 < Res < \infty$ & $0 < Rep < \infty$.This transformation maps $f(l,m)$ into a function $F(s,p)$ of the complex variables s and p . We extend the transform to a class of generalized functions [4].

2 TESTING FUNCTION SPACE $\mathfrak{LM}_{a,b,c,d}$ AND ITS DUAL $\mathfrak{LM}'_{a,b,c,d}$

Let us take an open set $\mathfrak{I}$ on positive real line. Let $\mathfrak{LM}_{a,b,c,d}$ is the space of all complex valued smooth function $\phi(l,m)$ defined on $\mathfrak{I}$, where $a,b,c,d,l,m \,\epsilon\, R^n$ and $s,p \,\epsilon\, C^n$, such that for each $\phi(l,m) \in \mathfrak{LM}_{a,b,c,d}$, we have

(2.1) $$\gamma_{j,k}\ \phi(l,m) \triangleq \sup_{\substack{-\infty<l<\infty \\ -\infty<m<\infty}} \left|\mathcal{J}_{a,b}(l)\, \lambda_{c,d}(m)\, m^{k+1}\, \mathfrak{D}_l^j\ \mathfrak{D}_m^k\ \phi(l,m)\right| < \infty$$

for each $j,k = 0,1,2,\ldots\ldots\ldots\ldots$ is bounded.

$$\text{where}\quad \mathcal{J}_{a,b}(l) \triangleq \begin{cases} e^{al} & 0 < l < \infty \\ e^{bl} & -\infty < l < 0 \end{cases}$$

$$\text{and}\quad \lambda_{c,d}(m) \triangleq \begin{cases} m^{-c} & 0 < m \le 1 \\ m^{-d} & 1 < m < \infty \end{cases}$$

$\mathcal{J}_{a,b}(l)$ and $\lambda_{c,d}(m)$ both denotes the spaces of all complex valued smooth functions $\phi(l,m)$ on $0 < l < \infty$ & $0 < m < \infty$ [8] .

Therefore, $\{\gamma_{j,k}\}_{\substack{j=0\\k=0}}^{\infty}$ is the collection of countable seminorm on the linear space $\mathfrak{LM}_{a,b,c,d}$. Again $\gamma_{0,0}$ is the norm on $\mathfrak{LM}_{a,b,c,d}$. Thus $\{\gamma_{j,k}\}_{\substack{j=0\\k=0}}^{\infty}$ is the countable multinorm on the linear space $\mathfrak{LM}_{a,b,c,d}$ which is called a countably multinormed space on $\mathfrak{I}$.

LEMMA 2.1 $\mathfrak{LM}_{a,b,c,d}$ IS A COMPLETE SPACE

For each non-negative numbers j and k , we take a Cauchy sequence $\{\phi_t\}_{t=1}^{\infty}$ in $\mathfrak{LM}_{a,b,c,d}$. Therefore, for every $t,\mu > H_{j,k}$, there exists a small arbitrary positive $\epsilon > 0$, such that

$$\gamma_{j,k}\left[\phi_t(l,m) - \phi_\mu(l,m)\right] < \epsilon$$
$$j,k = 0,1,2,\ldots\ldots\ldots\ldots$$

Consequently, we get

$$\left|m^{k+1}\,\mathfrak{D}_l^j\ \mathfrak{D}_m^k\left[\phi_t(l,m) - \phi_\mu(l,m)\right]\right| \le \epsilon$$

But there exists a smooth function $\phi(l,m)$ such that for each j,k,l and m ,we have the limit $\phi_\mu(l,m) \to \phi(l,m)$ as $\mu \to \infty$. So ,we get

(2.2) $$\left|m^{k+1}\,\mathfrak{D}_l^j\ \mathfrak{D}_m^k\left[\phi_t(l,m) - \phi(l,m)\right]\right| \le \epsilon$$
$$0 < l < \infty\ \ and\ 0 < m < \infty\ ,\ t > H_{j,k}$$

Thus for each j,k the seminorm $\gamma_{j,k}(\phi_t - \phi) \to 0$, as $t \to \infty$.

Also, we have

(2.3) $$\left|m^{k+1}\,\mathfrak{D}_l^j\ \mathfrak{D}_m^k\ \phi_t(l,m)\right| \le C$$

where C is a constant not depending upon t .

An appeal to (2.2) and (2.3) gives

$$\left|m^{k+1}\,\mathfrak{D}_l^j\ \mathfrak{D}_m^k\ \phi(l,m)\right|$$
$$= \left|m^{k+1}\,\mathfrak{D}_l^j\ \mathfrak{D}_m^k\left[\phi_t(l,m) + \phi(l,m) - \phi_t(l,m)\right]\right|$$

$$\leq \left| m^{k+1} \mathfrak{D}_l^j \ \mathfrak{D}_m^k \left[\phi(l,m) - \phi_t(l,m) \right] \right| + \left| m^{k+1} \mathfrak{D}_l^j \ \mathfrak{D}_m^k \phi_t(l,m) \right|$$

$$\leq \epsilon + C$$

Therefore, we get

$$(2.4) \qquad \left| m^{k+1} \mathfrak{D}_l^j \ \mathfrak{D}_m^k \ \phi(l,m) \right| \leq \epsilon + C$$

which shows that the limit function $\phi(l,m) \in \mathfrak{LM}_{a,b,c,d}$.

$\Rightarrow \mathfrak{LM}_{a,b,c,d}$ this is a complete space.

Therefore $\mathfrak{LM}_{a,b,c,d}$ is complete countably multinormed space on the open set $\mathfrak{I}$.

Let $\mathfrak{LM}'_{a,b,c,d}$ be the dual of $\mathfrak{LM}_{a,b,c,d}$. Thus $f \in \mathfrak{LM}'_{a,b,c,d}$ if it is a continuous a linear functional on $\mathfrak{LM}_{a,b,c,d}$. Since $\mathfrak{LM}_{a,b,c,d}$ is a testing- function space [9] . So we say $\mathfrak{LM}'_{a,b,c,d}$ is the space of generalized functions which is also a complete space due to [9]. Thus for any $f \in \mathfrak{LM}'_{a,b,c,d}$ and $\phi \in \mathfrak{LM}_{a,b,c,d}$, the generalized function is denoted as $< f, \phi >$.

LEMMA 2.2 TO PROVE THAT $e^{-sl}\, m^{p-1} \in \mathfrak{LM}_{a,b,c,d}$

Let $m^{k+1} \mathfrak{D}_l^j \ \mathfrak{D}_m^k \left[e^{-sl} \, m^{p-1} \right] = P_{jk}(l,m)$

where $P_{jk}(l,m)$ be the polynomials in l,m for all $j,k = 0,1,2,\dots,$ $0 < l,m < \infty$ and $0 < Res < \infty$ and $0 < Rep < \infty$.

Therefore, we get

$$\sup_{\substack{0<l<\infty \\ 0<m<\infty}} \left| m^{k+1} \mathfrak{D}_l^j \ \mathfrak{D}_m^k \left[e^{-sl} \, m^{p-1} \right] \right| \text{ is bounded}$$

for each $0 < s < \infty$ *and* $0 < p < \infty$ $j, k = 0, 1, 2, \dots\dots\dots\dots$

Hence, $\Rightarrow e^{-sl} \, m^{p-1} \in \mathfrak{LM}_{a,b,c,d}$

LEMMA 2.3 : D($\mathfrak{I}$) IS A SUBSPACE OF $\mathfrak{LM}_{a,b,c,d}$

Let $\phi \in D(\mathfrak{I}) \Rightarrow \sup_{l,m \in \mathfrak{I}} \left| \mathfrak{D}_l^j \ \mathfrak{D}_m^k \ \phi(l,m) \right|$ is bounded

where $\phi(l,m)$ is a complex valued smooth function nonzero within the compact set K of $\mathfrak{I} =]0,\infty[$ and zero outside K.

$$\Rightarrow \sup_{\substack{0<l<\infty \\ 0<m<\infty}} \left| m^{k+1} \mathfrak{D}_l^j \ \mathfrak{D}_m^k \ \phi(l,m) \right| \text{ is}$$

bounded $\Rightarrow \phi \in \mathfrak{LM}_{a,b,c,d}$

$$\Rightarrow D(\mathfrak{I}) \subseteq \mathfrak{LM}_{a,b,c,d}$$

From the above relation ,we find a convergent sequence in D($\mathfrak{I}$) implies the sequence also converges in $\mathfrak{LM}_{a,b,c,d}$. Consequently, the restriction of $f \in \mathfrak{LM}'_{a,b,c,d}$ to D($\mathfrak{I}$) is in D′($\mathfrak{I}$) . However, D($\mathfrak{I}$) is not dense in $\mathfrak{LM}_{a,b,c,d}$. Thus we cannot identify $\mathfrak{LM}'_{a,b,c,d}$ with a subspace of D′($\mathfrak{I}$). Actually different members of $\mathfrak{LM}'_{a,b,c,d}$ can be found whose restriction to D($\mathfrak{I}$) are identical.

LEMMA 2.4 : $\mathfrak{LM}_{a,b,c,d}$ IS A DENSE SUBSPACE OF E($\mathfrak{I}$)

Let $\phi \in \mathfrak{LM}_{a,b,c,d} \Rightarrow \sup_{\substack{0<l<\infty \\ 0<m<\infty}} \left| m^{k+1} \mathfrak{D}_l^j \ \mathfrak{D}_m^k \ \phi(l,m) \right|$ is bounded, where $j,k = 0,1,2,\dots$

$$\Rightarrow \sup_{l,m \in S} \left| \mathfrak{D}_l^j \ \mathfrak{D}_m^k \ \phi(l,m) \right| \text{ is bounded,}$$

where S is a compact set of $\mathfrak{I} =]0,\infty[$.

$$\Rightarrow \phi \in E(\mathfrak{I})$$

Therefore, we get $\mathfrak{LM}_{a,b,c,d} \subseteq E(\mathfrak{I})$.

We also get form the above lemma $D(\mathfrak{I}) \subseteq \mathfrak{LM}_{a,b,c,d} \subseteq E(\mathfrak{I})$. Also $D(\mathfrak{I})$ is a dense subspaces of $E(\mathfrak{I})$. It follows that $\mathfrak{LM}_{a,b,c,d}$ is a dense subspace of $E(\mathfrak{I})$. Hence we get the result.

3 EXTENSION OF $\mathfrak{LM}_{a,b,c,d}$ TRANSFORM TO A CLASS OF GENERALIZED FUNCTIONS

Let function f be a $\mathfrak{LM}_{a,b,c,d}$-transformable generalized function if it satisfied following property :

1. f is a functional on some domain $\eth(f)$ of conventional functions.
2. f is additive in the sense that if $\phi, \theta, \phi + \theta$ are all members of $\eth(f)$, then $< f, \phi + \theta > = < f, \phi > + < f, \theta >$ is in $\mathfrak{LM}_{a,b,c,d}$.
3. $\mathfrak{LM}_{a,b,c,d} \subset \eth(f)$ the restriction of f to $\mathfrak{LM}_{a,b,c,d}$ is in $\mathfrak{LM}'_{a,b,c,d}$.

Since, $e^{-sl}\, m^{p-1} \in \mathfrak{LM}_{a,b,c,d}$ for $0 < l < \infty$ & $0 < m < \infty$ and $0 < Res < \infty$ & $0 < Rep < \infty$; We define the generalized $\mathfrak{LM}_{a,b,c,d}$–transform of f by

$$F(s,p) = \langle f(l,m),\ e^{-sl}\, m^{p-1} \rangle$$

for $s,p \in \Omega f$ and $\Omega f = \{s,p \colon 0 < Res, Rep < \infty\}$.

Ωf is called the region of definition for $\mathfrak{LM}_{a,b,c,d}$-transform and (0,∞) are the abscissa of definition.

4 ANALYTICITY THEOREM OF $\mathfrak{LM}_{a,b,c,d}$ TRANSFORM

STATEMENT: $F(s,p) = \langle f(l,m),\ e^{-sl}\, m^{p-1} \rangle$ for $s,p \in \Omega f$ and $\Omega f = \{s,p \colon 0 < Res < \infty \ \& \ 0 < Rep < \infty\}$, then $F(s,p)$ is analytic on Ωf and $\mathfrak{D}_s \mathfrak{D}_p \, F(s,p) = \langle f(l,m), \frac{\partial}{\partial s}\frac{\partial}{\partial p}\ e^{-sl}\, m^{p-1} \rangle$.

PROOF : Let s and p be two arbitrary member . Δs and Δp are respectively very small complex number respectively on s and p , such that $|\Delta s| \to 0$ and $|\Delta p| \to 0$.

$$F(s,p) = \langle f(l,m),\ e^{-sl}\, m^{p-1} \rangle$$

$$F(s+\Delta s, p+\Delta p) = \langle f(l,m),\ e^{-(s+\Delta s)l}\, m^{(p+\Delta p)-1} \rangle$$

Therefore we have

$$\frac{F(s+\Delta s, p+\Delta p) - F(s,p)}{\Delta s\, \Delta p} = \langle f(l,m), \frac{\partial}{\partial s}\ e^{-sl}\ \frac{\partial}{\partial p}\ m^{p-1} \rangle = \langle f(l,m), \psi_{\Delta s\, \Delta p} \rangle$$

where,

$$\psi_{\Delta s\, \Delta p}(l,m) = \frac{\partial}{\partial s}\ e^{-sl}\ \frac{\partial}{\partial p}\ m^{p-1}$$

$$= \frac{1}{\Delta s}\left\{\left[e^{-(s+\Delta s)l} - e^{-sl}\right] - \frac{\partial}{\partial s} e^{-sl}\right\} \frac{1}{\Delta p}\left\{\left[m^{(p+\Delta p)-1} - m^{p-1}\right] - \frac{\partial}{\partial p} m^{p-1}\right\}$$

Now, we have to show that $\psi_{\Delta s\, \Delta p} \to 0$ in $\mathfrak{LM}_{a,b,c,d}$ as $|\Delta s| \to 0$ and $|\Delta p| \to 0$. Since $f(l,m) \in \mathfrak{LM}'_{a,b,c,d}$, so this will imply that $\langle f(l,m), \psi_{\Delta s\, \Delta p} \rangle \to 0$ as $|\Delta s| \to 0$ and $|\Delta p| \to 0$. For, let C_1 and C_2 denote the circle with center at s and p respectively with r_1 and r_2 , we get

$(-\mathfrak{D}_l)^j (\mathfrak{D}_m)^k \ \psi_{\Delta s\, \Delta p}$

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$$= \frac{1}{\Delta s}\left\{ \mathfrak{D}_l^j \left[e^{-(s+\Delta s)l} - e^{-sl}\right] - \frac{\partial \mathfrak{D}_l^j}{\partial s} e^{-sl}\right\} \frac{1}{\Delta p}\left\{ \mathfrak{D}_m^k \left[m^{(p+\Delta p)-1} - m^{p-1}\right] - \frac{\partial \mathfrak{D}_m^k}{\partial p} m^{p-1}\right\}$$

(Interchanging the differentiation on s and p with the differentiation on l and m)

$$\psi_{\Delta s\,\Delta p} = \frac{1}{\Delta s}\left\{\left[U_j(s+\Delta s, l) - U_j(s,l)\right] - \frac{\partial}{\partial s} U_j(s,l)\right\} * \frac{1}{\Delta p}\left\{\left[V_k(p+\Delta p, m) - V_k(p,m)\right] - \frac{\partial}{\partial p} V_k(p,m)\right\}$$

Where $U_j(s+\Delta s, l)$ and $U_j(s,l)$ are polynomials in l such that $U_j(s,l) \to U_j(s+\Delta s, l)$ as $s \to s+\Delta s$ .Similarly $V_k(p+\Delta p, m)$ and $V_k(p,m)$ are polynomials in m such that $V_k(p,m) \to V_k(p+\Delta p, m)$as $\to p+\Delta p$.

By Cauchy's integral formula , we get

$$\mathfrak{D}_l^j\, \mathfrak{D}_m^k\ \psi_{\Delta s\,\Delta p}(l,m) = \frac{1}{2\pi i \Delta s}\left\{\int_{C_1}\left[\frac{1}{\varrho - s - \Delta s} - \frac{1}{\varrho - s}\right] U_j(\varrho, l)\, d\varrho - \int_{C_1}\left[\frac{1}{(\varrho - s)^2}\right] U_j(\varrho, l)\, d\varrho\right\} * \frac{1}{2\pi i \Delta p}\left\{\int_{C_2}\left[\frac{1}{\sigma - p - \Delta p} - \frac{1}{\sigma - p}\right] V_k(\sigma, m)\, d\sigma - \int_{C_2}\left[\frac{1}{(\sigma - p)^2}\right] V_k(\sigma, m)\, d\sigma\right\}$$

Where $\varrho \in C_1$ and $\sigma \in C_2$,the respectively circles with centers s & p and radius r_1 & r_2. Therefore, we get

$$\mathfrak{D}_l^j\, \mathfrak{D}_m^k\ \psi_{\Delta s\,\Delta p} = \frac{1}{2\pi i \Delta s}\left\{\int_{C_1}\left[\frac{1}{\varrho - s - \Delta s} - \frac{1}{\varrho - s} - \frac{\Delta s}{(\varrho - s)^2}\right] U_j(\varrho, l)\, d\varrho\right\} * \frac{1}{2\pi i \Delta p}\left\{\int_{C_2}\left[\frac{1}{\sigma - p - \Delta p} - \frac{1}{\sigma - p} - \frac{\Delta p}{(\sigma - p)^2}\right] V_k(\sigma, m)\, d\sigma\right\}$$

$$= \frac{\Delta s\, \Delta p}{(2\pi i)^2}\left\{\int_{C_1}\left[\frac{1}{(\varrho - s - \Delta s)(\varrho - s)^2}\right] U_j(\varrho, l)\, d\varrho\right\} * \left\{\int_{C_2}\left[\frac{1}{(\sigma - p - \Delta p)(\sigma - p)^2}\right] V_k(\sigma, m)\, d\sigma\right\}$$

Let $|\varrho - s| = R_1, |\varrho - s - \Delta s| = r_1, |\sigma - p - \Delta p| = R_2$ & $|\sigma - p - \Delta p| = r_2$ ($r_1 < R_1$ and $r_2 < R_2$). Since $s \in C_1$ & $p \in C_2$ and $0 < l < \infty$ & $0 < m < \infty$.

So, $\left|\mathcal{J}_{a,b}(l)\, \lambda_{c,d}(m)\, m^{k+1}\, U_j(\varrho, l)\, V_k(\sigma, m)\right| \le M$

Thus, we get

$$\left|\mathcal{J}_{a,b}(l)\, \lambda_{c,d}(m)\, m^{k+1}\, \mathfrak{D}_l^j\ \mathfrak{D}_m^k\ \psi_{\Delta s\,\Delta p}(l,m)\right| \le \frac{K}{(2\pi)^2}|\Delta s||\Delta p| \frac{1}{(R_1\, R_2)^2 r_1 r_2}\int_{C_1}|d\varrho| \int_{C_2}|d\sigma|$$

$$\le \frac{K}{(2\pi)^2}|\Delta s||\Delta p| \frac{1}{(R_1\, R_2)^2 r_1 r_2}\ 2\pi R_1\ 2\pi R_2$$

$$\le |\Delta s||\Delta p| \frac{K}{R_1\, R_2 r_1 r_2}$$

From the above equation, it follows that $\psi_{\Delta s\,\Delta p} \to 0$ in $\mathfrak{LM}_{a,b,c,d}$ as $|\Delta s| \to 0$ and $|\Delta p| \to 0$.This completes the proof.

5 OPERATION - TRANSFORM FORMULA OF $\mathfrak{LM}_{a,b,c,d}$ TRANSFORM

5.1 DIFFERENTIATION

If $\phi(l,m) \in \mathfrak{LM}_{a,b,c,d}$ where $\mathfrak{LM}_{a,b,c,d}$ is the space of all complex valued smooth functions (l,m) . Such that for $\phi(l,m) \in \mathfrak{LM}_{a,b,c,d}$, we have

$$\gamma_{j,k}\ \phi(l,m) \triangleq \sup_{\substack{-\infty<l<\infty \\ -\infty<m<\infty}} \left|e^{-sl}\, m^{p-1}\, \mathfrak{D}_l^j\ \mathfrak{D}_m^k\ \phi(l,m)\right| < \infty$$

for each $j, k = 0, 1, 2, \dots\dots\dots$ is bounded.

We shall prove that

$$\gamma_{j,k}\left[-\mathfrak{D}_l\, \mathfrak{D}_m\, \phi(l,m)\right] = \gamma_{j+1,k+1}\left[\phi(l,m)\right]$$

PROOF:- It is easy to say that $\phi \to -\mathfrak{D}\phi$ is a continuous and linear mapping of $\mathfrak{LM}_{a,b,c,d}$ onto itself. Therefore the adjoint mapping $f \to \mathfrak{D}f$ is also a continuous and linear mapping of $\mathfrak{LM}'_{a,b,c,d}$ onto itself where $\mathfrak{LM}'_{a,b,c,d}$ is the dual of $\mathfrak{LM}_{a,b,c,d}$ (Zemanian [9]) ,we get

$$\langle \mathfrak{D}_l\, \mathfrak{D}_m f(l,m), \phi(l,m)\rangle = \langle f(l,m), (-\mathfrak{D}_l)(-\mathfrak{D}_m)\phi(l,m)\rangle = \langle f(l,m), s(p-1)e^{-sl}\, m^{p-2}\rangle$$

Similarly, we also define a mapping $f \to \mathfrak{D}_l^j\ \mathfrak{D}_m^k\ f$ is also a continuous and linear mapping of $\mathfrak{LM}'_{a,b,c,d}$.

$$\langle\, \mathfrak{D}_l^j\ \mathfrak{D}_m^k\ f(l,m), \phi(l,m)\rangle = \langle f(l,m), (-\mathfrak{D}_l)^j\, (-\mathfrak{D}_m)^k\, \phi(l,m)\rangle = \langle f(l,m), s^j\ (p-1)_k\ e^{-sl}\ m^{p-k-1}\rangle$$

for each $j, k = 0, 1, 2, \dots\dots\dots$

where $(a)_k = a\,(a+1)(a+2)\dots\dots\dots(a+k-1) \qquad k = 1,2,3, \dots\dots\dots$

Therefore , we get

$$\gamma_{j,k}\left[\,\phi(l,m)\right] = \mathfrak{LM}\, \mathfrak{D}_l^j\ \mathfrak{D}_m^k\ f = s^j\ (p-1)_k\, F(s, p-k) \qquad s, p-k \in \Omega f$$

Again differentiating, we get

$$\Rightarrow \quad \gamma_{j,k}\left[-\mathfrak{D}_l\, \mathfrak{D}_m\, \phi(l,m)\right] = s^{j+1}\ (p-1)_{k+1}\, F(s, p-(k+1)) = \gamma_{j+1,k+1}\left[\phi(l,m)\right]$$

Hence we get the result.

5.2 MULTIPLICATION BY AN EXPONENTIAL AND POWER OF m

Let α, β be two complex number , we shall prove $\phi(l,m) \to e^{-sl}\, m^{p-1}\phi(l,m)$ is a continuous and linear mapping of $\mathfrak{LM}_{a,b,c,d}$ on to itself .

$$\langle f(l,m), e^{-\alpha l}\, m^{\beta}\, \phi(l.m)\rangle = F(s+\alpha, p+\beta)$$

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PROOF :- Let $\phi(l,m) \in \mathfrak{LM}_{a,b,c,d}$, we have

$$\mathfrak{D}_l^j \ \mathfrak{D}_m^k \left[e^{-\alpha l} \, m^\beta \, \phi(l.m)\right]$$

$$= \sum_{r=0}^{j} \sum_{t=0}^{k} {}^jC_r \ {}^kC_t \ \mathfrak{D}_l^{j-r} e^{-\alpha l} \ \mathfrak{D}_m^{k-t} \ m^\beta \ \mathfrak{D}_l^r \ \mathfrak{D}_m^t \ \phi(l,m)$$

$$= \sum_{r=0}^{j} \sum_{t=0}^{k} {}^jC_r \ {}^kC_t \ (-\alpha)^{j-r} \ e^{-\alpha l} \ \beta(\beta-1) \ldots\ldots\ldots (\beta - t + 1) \ m^{\beta-t+1} \, \mathfrak{D}_l^r \ \mathfrak{D}_m^t \ \phi(l,m)$$

Therefore , we get

$$\left| \ {}^jC_r \ {}^kC_t \ (-\alpha)^{j-r} \ e^{-\alpha l} \ \beta(\beta-1) \ldots\ldots\ldots (\beta - t + 1) \ m^{\beta-t+1} \right| \le \mathfrak{C}$$

Thus , we get

$$\gamma_{j,k} \left[e^{-\alpha l} \, m^\beta \, \phi(l.m)\right] \le \mathfrak{C} \sum_{r=0}^{j} \sum_{t=0}^{k} \gamma_{r,t} \ \left[\phi(l,m)\right]$$

It follows that $\phi(l,m)$ is a continuous and linear mapping of $\mathfrak{LM}_{a,b,c,d}$ onto itself . Therefore, the adjoint mapping $f(l,m) \to e^{-\alpha l} \, m^\beta \, f(l,m)$ of $\phi(l,m) \to e^{-\alpha l} \, m^\beta \phi(l,m)$, is also continuous and linear mapping of $\mathfrak{LM}'_{a,b,c,d}$ onto itself due to theorem of Zemanian , where $\mathfrak{LM}'_{a,b,c,d}$ is the dual of $\mathfrak{LM}_{a,b,c,d}$, we get

$$\langle e^{-\alpha l} \, m^\beta \, f(l,m), \phi(l,m)\rangle = \langle f(l,m), e^{-\alpha l} \, m^\beta \, \phi(l.m)\rangle$$
$$= \langle f(l,m), e^{-(s+\alpha) l} \, m^{p+\beta-1} \rangle$$

From definition of generalized function

$$\mathfrak{LM}\left[e^{-\alpha l} \, m^\beta \, f(l,m)\right] = F(s+\alpha, p+\beta) \qquad s+\alpha, p+\beta \in \Omega f$$

Hence we get the result.

5.3 MULTIPLICATION BY AN POWER OF l AND log m

Let α and β be two real numbers , such that $\alpha, \beta \ge 0$. We shall prove $\phi(l,m) \to l^\alpha \, (\log m)^\beta \ \phi(l,m)$ is a continuous and linear mapping of $\mathfrak{LM}_{a,b,c,d}$ onto itself .

PROOF :- Let $\phi(l,m) \in \mathfrak{LM}_{a,b,c,d}$, we have

$$\mathfrak{D}_l^j \ \mathfrak{D}_m^k \left[l^\alpha \, (\log m)^\beta \ \phi(l.m)\right]$$

$$= \sum_{r=0}^{j} \sum_{t=0}^{k} {}^jC_r \ {}^kC_t \ \mathfrak{D}_l^{j-r} \, l^\alpha \ \mathfrak{D}_m^{k-t} \ (\log m)^\beta \ \mathfrak{D}_l^r \ \mathfrak{D}_m^t \ \phi(l,m)$$

Therefore , we get

$$\left| \mathfrak{D}_l^j \ \mathfrak{D}_m^k \left[l^\alpha \, (\log m)^\beta \ \phi(l.m)\right]\right| \le \mathcal{K} \sum_{r=0}^{j} \sum_{t=0}^{k} \left| \mathfrak{D}_l^j \, \mathfrak{D}_m^k \ \phi(l,m)\right|$$

$$\gamma_{j,k} \left[l^\alpha \, (\log m)^\beta \ \phi(l.m)\right] \le \mathcal{K} \sum_{r=0}^{j} \sum_{t=0}^{k} \gamma_{r,t} \ \left[\phi(l,m)\right]$$

It follows that $\phi(l,m) \to l^\alpha \, (\log m)^\beta \, \phi(l,m)$ is a continuous and linear mapping of $\mathfrak{LM}_{a,b,c,d}$ onto itself . Therefore, from the theorem of Zemanian we define a adjoint mapping $f(l,m) \to l^\alpha \, (\log m)^\beta \ f(l,m)$ of is also continuous and linear mapping of $\mathfrak{LM}'_{a,b,c,d}$ onto itself , where $\mathfrak{LM}'_{a,b,c,d}$ is the dual of $\mathfrak{LM}_{a,b,c,d}$, we get

$$\langle l \, (\log m) f(l,m), \phi(l,m)\rangle = \langle f(l,m), l \, (\log m) \phi(l.m)\rangle$$
$$= \langle f(l,m), l \, (\log m) e^{-sl} \, m^{p-1} \rangle$$
$$= (-\mathfrak{D}_s)(\mathfrak{D}_p)\langle f(l,m), e^{-sl} \, m^{p-1} \rangle$$

If f be a generalized function of $\mathfrak{LM}_{a,b,c,d}$, then we get

$$\langle l^\alpha \, (\log m)^\beta \ f(l,m), \phi(l,m)\rangle$$
$$= \langle f(l,m), l^\alpha \, (\log m)^\beta \ \phi(l.m)\rangle$$
$$= \langle l^{\alpha-1} \, (\log m)^{\beta-1} \ f(l,m), l \, e^{-sl} \, (\log m) \ m^{p-1} \rangle$$
$$= (-\mathfrak{D}_s)\left(\mathfrak{D}_p\right)\langle l^{\alpha-1} \, (\log m)^{\beta-1} \ f(l,m), e^{-sl} \ m^{p-1} \rangle$$
$$= (-\mathfrak{D}_s)\left(\mathfrak{D}_p\right)\langle l^{\alpha-2} \, (\log m)^{\beta-2} \ f(l,m), l e^{-sl} (\log m) \ m^{p-1} \rangle$$
$$= (-\mathfrak{D}_s)^2\left(\mathfrak{D}_p\right)^2\langle l^{\alpha-2} \, (\log m)^{\beta-2} \ f(l,m), e^{-sl} \ m^{p-1} \rangle$$
$$= \cdots = (-\mathfrak{D}_s)^\alpha\left(\mathfrak{D}_p\right)^\beta\langle f(l,m), e^{-sl} \ m^{p-1} \rangle$$

From definition of generalized function

$$\mathfrak{LM}\left[\, l^\alpha \, (\log m)^\beta \ f(l,m)\right] = (-\mathfrak{D}_s)^\alpha\left(\mathfrak{D}_p\right)^\beta F(s,p) \ \ s,p \in \Omega f$$

Hence we get the result.

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