# Congruence on Left Regular Bands of Groups of Left Quotients

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**Abstract:** We provide a complete algebraic structure for Left regular bands of groups of left quotients Q by constructing congruence for it. Our construction uses the structural properties of the semigroup which makes it defer from the known congruence construction methods. We also deduce some properties arising from the said construction.

**AMS Classification No:** 20M10

**Key Words:** Congruence, Left Orders, Green's Relations.

## 1.0 Introduction

Left regular bands of groups of left quotients Q, was studied by [1]. He developed a structure for Q. Here our Q is $\mathcal{R}$-unipotent and is also semigroup of left quotients of the semigroup S. Note that a regular semigroup in which its set of idempotent is a left regular band such that for any idempotent e , f in S, efe = ef is called $\mathcal{R}$ -unipotent(or right inverse) semigroup. We recall that a semigroup Q containing a subsemigroup S is said to be a semigroup of left quotients of S if every element q in Q can be written as $q = a^{-1}b$ for some elements a, b in S with $a^2 \neq 0$ and $a^{-1}$ is the inverse of a in the group $\mathcal{H}$-class $H_a$ of Q. In this case S is called a left order in Q. A lot of Authors including [2] and [8] have studied and characterized $\mathcal{R}$-unipotent semigroups which are bands of groups.

Our main interest in the paper is to construct congruence for this semigroup. This would provide a complete algebraic structure for it. So, in section 2, we provide basic definitions and concepts that would enable the reader to follow our discussion. In section 3, we give a brief algebraic structure of the semigroup in discussion. The main construction of the congruence is in section 4, while section 5 deals with the normal subgroup and other properties derived from the constructed structure.

## 2.0 Preliminary.

We shall start this section by defining some basic concept that would be very useful in explaining the structure of left regular bands of groups of left quotients, Q. The reader should refer to [3] and [6] for concepts clarification.

### 2.1 Definition

ISSN 2229-5518

(i) A semigroup S is called $\mathcal{R}$-unipotent or left inverse semigroup if every $\mathcal{R}$-class contains a unique idempotent.

(ii) Let S be a semigroup. Then S is called a union of groups if every element of S is contained in a subgroup of S.

(iii) A band is a semigroup in which every element is idempotent.

(iv) A left regular band of groups is a band such that for e, f $\in$ E, efe = ef, E is a set of idempotents in the semigroup S.

**2.2 Remark**

(i) In a union of groups, every $\mathcal{H}$-class is a group.

(ii) A band is an example of a semigroup that is a union of groups since each element constitutes a group all on its own. The following properties of band will be very useful in our later work.

**2.3 Theorem [6]**

A band is a semilattice of rectangular bands.

**2.4 Proposition**

*If S is a semigroup, then the following statements are equivalent;*

(i) S is a rectangular band.

(ii) $(\forall a, b \in S)$ ab = ba $\Rightarrow$ a = b

(iii) $(\forall a, b \in S)$ aba = a

(iv) $(\forall a \in S)$ $a^2 = a$ and $(\forall a, b, c \in S)$ abc = ac

**3.0 Structure of Left Regular Bands of Groups of Left Quotients**

Suppose Q is a band Y of groups $G_\alpha$, $\alpha \in$ Y, where for any $\alpha, \beta \in Y$, $G_\alpha \cap G_\beta = \varnothing$, if $\alpha \neq \beta$ and Q = $\bigcup_{\alpha \varepsilon y} G_\alpha$ ; $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ , that is, Y is a left regular band.

We note that the groups $G_\alpha$, $\alpha \in Y$ are taken to be the $\mathcal{H}$-classes of Q. Note also that if b $\in S_\alpha$ and c $\in S_\beta$, then bc $\in S_{\alpha\beta}$ and since Y is a left regular band, bcb $\in S_\alpha S_\beta S_\alpha \subseteq S = S_{\alpha\beta\alpha} = S_{\alpha\beta}$. By right reversibility of $S_{\alpha\beta}$, $\exists$ x', y' $\in S_{\alpha\beta}$ with x'bcb = y'bc. Putting x = x'bc, y = y'b, we have, xb = yc, where x = x'bc $\in S_{\alpha\beta}S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and y = y'b $\in$ S. Furthermore, for any a $\in S_\alpha$, xa $\in S_\alpha = S_{\alpha\beta}$, yd $\in S_{\alpha\beta\beta} = S_{\alpha\beta}$ and so, $(xa)^{-1}$ yd exists in $G_{\alpha\beta}$. With the above in view, we now define product on Q by

$$a^{-1}b \,.\, c^{-1}d = (xa)^{-1}yd$$

where, if a, b $\in S_\alpha$; c, d $\in S_\beta$, then x, y $\in S_{\alpha\beta}$, are chosen so that xb = yc. Now, since Y is a left regular band, xa $\in S_{\alpha\beta}S_\alpha \subseteq S_{\alpha\beta\alpha} = S_{\alpha\beta}$ and $(xa)^{-1} \in G_{\alpha\beta}$. Also, yd $\in S_{\alpha\beta}S_\beta \subseteq S_{\alpha\beta} = S_{\alpha\beta}$ and yd $\in G_{\alpha\beta}$. Thus, the product $(xa)^{-1}$ yd is taken as the product in $G_{\alpha\beta}$. The property of left regularity of the band Y together with right reversibility and cancellativity of $S_\alpha$, $\alpha \in Y$ makes Q under the defined multiplication a groupiod.

Let us recall some useful properties of the semigroup S. If $\alpha \in Y$ and a, b are elements of $S_\alpha$, then a$\mathcal{R}^*$b in S. Also for any element a $\in$ S, a$\mathcal{H}^*a^2$. These properties make Q similar to the Clifford semigroup of left quotient and so

ISSN 2229-5518

using [4], we conclude that Q is associative. The product in Q is an extension of that in S. So for $G_\alpha$, $G_\beta$ in Q, we have, $G_\alpha G_\beta \subseteq G_{\alpha\beta}$, for any $\alpha$, $\beta \in Y$. Therefore, Q is a left regular band of groups $G_\alpha$, $\alpha \in Y$. This construction shows that Q is a semigroup of left quotients of S. Therefore; a semigroup S has a left regular band of groups as a semigroup of left quotients if and only if S is a left regular band of right reversible, cancellable semigroups.

**3.1 Note**

(i) Suppose S is a left regular band of right reversible, cancellable semigroups, then for any decomposition of S as a left regular band Y of right reversible, cancellative semigroup, we can construct a semigroup Q of left quotient of S, where Q is a left regular b and Y of groups.

(ii) Neither the decomposition of S nor the construction of Q is unique. To overcome this problem of uniqueness in the inverse semigroup case, the notion of stratified semigroup of left quotient becomes very relevant.

(iii) Let Q be an over semigroup of a semigroup S. Then Q is a stratified semigroup of left quotient of S. if

(a) for any a, b of S, $a\mathcal{R}b$ inQ if and only if $a\mathcal{R}^*b$ in S; $a\mathcal{L}b$ in Q if and only if $a\mathcal{L}^*b$ in S.

(b) every element a of S is in a subgroup of Q whenever, $a\mathcal{H}^*a^2$ in S.

(c) every element of Q can be written as $a^{-1}b$, where a, b $\in$ S, and $a\mathcal{R}b$ in Q.

**3.2 Definition**

*Let S be a subsemigroup of Q, we say that S is a normal subsemigroup of Q if:*

(i) $E_Q \subseteq S$

(ii) $b^{-1}aSa^{-1}b \subseteq S$, for $a^{-1}b \in Q$.

**4.0 The Congruence**

We shall in this section, construct normal and reversible congruences using the structural properties of Q. This will confirm the fact that in the quotient considered there is always there always a reversible congruence. This is the crust of the work we have been doing.

**4.1 Definition**

(i) A congruence $\rho$ on $E_Q$ is called a normal congruence if for any $e_\alpha$, $f_\beta \in E_Q$ and $a^{-1}b \in Q$, we have, $e_\alpha \rho f_\beta$ implies $b^{-1}ae_\alpha a^{-1}b\rho b^{-1}af_\beta a^{-1}b$.

(ii) A congruence pair for Q is a pair (S, $\rho$), where S is a normal subsemigroup of Q and $\rho$ a normal congruence on $E_Q$, satisfying.

(i) $a^{-1}be_\alpha \in S$, $e_\alpha \rho b^{-1}aa^{-1}b$ implies $a^{-1}b \in S$ ($a^{-1}b \in Q$ and $e_\alpha \in E_Q$)

(ii) $m^{-1}n \in S$ implies $m^{-1}nn^{-1}m\rho n^{-1}mm^{-1}n$.

**4.2 Remark**

In the later part of the work, we will construct a normal subgroup of Q, which will coincide with the definition above. The concept above could be used to characterize the properties of Q. We have however avoided using the concept of trace, congruence pair, lattices of congruence and kernel normal system to characterize Q. We consider this as areas of future expansion of the work.

**4.3** Let Q be a left regular bands of groups of left quotients of the subsemigroup S. We note that idempotents in Q are denoted in the form $aa^{-1}$ or $a^{-1}a$. Also note that inverse of the element $a^{-1}b$ in the subgroup of Q is represented by $b^{-1}a$. In what follows, the structure of Q as stated above is still maintained.

Now define for elements $a^{-1}b$, $c^{-1}d$ in Q a relation ρ on in the following ways: $a^{-1}b\rho c^{-1}d$ if for idempotents $e_\alpha$, $f_\beta$ in we have $b^{-1}ae_\alpha a^{-1}b\rho c^{-1}df\rho d^{-1}c$. That is, for $a^{-1}b$, $c^{-1}d$, $e_\alpha$, $f_\beta$ in Q, $a^{-1}b\rho c^{-1}d$ implies $b^{-1}ae_\alpha a^{-1}b\rho c^{-1}df_\beta d^{-1}c$. We shall show that the relation ρ is congruence. Let us start by show that it is first an equivalent relation. Note that $a^{-1}b = b^{-1}aa^{-1}aa^{-1}ba^{-1}b$. So, if $a^{-1}bpa^{-1}b$ then $b^{-1}aa^{-1}aa^{-1}ba^{-1}b\rho b^{-1}aa^{-1}aa^{-1}ba^{-1}b$. That is, ρ is reflexive. Now suppose, $a^{1-}b\rho c^{-1}d$ implies $b^{-1}ae_\alpha a^{-1}b\rho c^{-1}df_\beta d^{-1}c$. By reflexivity of ρ, $a^{-1}b\rho a^{-1}b$ and $c^{-1}d\rho c^{-1}d$. Combining, we have, $a^{-1}bc^{-1}d\rho a^{-1}bc^{-1}d$. That is $a^{-1}b\ (c^{-1}d\rho a^{-1}b)\ c^{-1}d$. That is, $c^{-1}dpa^{-1}b$. So, by definition, since $c^{-1}d\rho a^{-1}b$ then $d^{-1}cec^{-1}d\rho a^{-1}bf_\beta b^{-1}a$. Thus, ρ is symmetly. We show that ρ is transitive. Let $a^{-1}b\rho c^{-1}d$ and $c^{-1}d\rho m^{-1}n$, then $b^{-1}ae_\alpha a^{-1}b\rho c^{-1}df_\beta d^{-1}c$ and $d^{-1}ce_\alpha c^{-1}d\rho m^{-1}n\ g_\delta n^{-1}m$, where $e_\alpha$, $f_\beta$, $g_\delta$ are idempotents in Q. Then, $b^{-1}ae_\alpha a^{-1}b\rho c^{-1}df_\beta d^{-1}c$ and $d^{-1}c^{-1}ce_\alpha c^{-1}d\rho m^{-1}n\ g_\delta n^{-1}m$. So, $b^{-1}ae_\alpha a^{-1}bpc^{-1}df_\beta d^{-1}c = d^{-1}ce_\alpha c^{-1}d\rho m^{-1}n\ g_\delta n^{-1}m$. That is, $b^{-1}ae_\alpha a^{-1}b\rho c^{-1}df_\beta d^{-1}c\rho m^{-1}n\ g_\delta n^{-1}m$, that is, $b^{-1}ae_\alpha a^{-1}b\rho m^{-1}n\ g_\delta n^{-1}m$. This is possible only if $a^{-1}b\rho m^{-1}n$. That is, Let $a^{-1}b\rho c^{-1}d$ and $c^{-1}d\rho m^{-1}n$ implies $a^{-1}b\rho m^{-1}n$. Therefore, ρ is transitive. We have thus shown that

In Q there is a normal congruence.

**4.4 Theorem**

*Let Q be a left regular band of groups of left quotient of the subsemigroup S, then, there is always a right reversible.*

Proof

On Q, we define the relation ρ as follows:

For $a^{-1}b, c^{-1}d \in Q$, $a^{-1}b\ \rho\ c^{-1}d$ if and only if there exists $x, y \in S_{\alpha\beta}$ such that $xb = yc$, where $a, b \in S_\alpha$ and $c, d \in S_\beta$. The existence of x, y in $S_{\alpha\beta}$ is guaranteed by the structure of Q as given above.

We now show that the relation p is an equivalence relation. Let $a^{-1}b \in Q$. Suppose that $a^{-1}b \, \rho \, c^{-1}d$, where a, b ∈ S. Note that in Q, $S_\alpha S_\alpha = S_\alpha^2 = S_\alpha$. Now $S_\alpha$ is right reversible, so there exists

X, $y \in S_\alpha = S_\alpha^2$ such that xb = ya. So, ρ is reflexive. Let $a^{-1}b\rho c^{-1}d$ if and only if there exists x, $y \in S_{\alpha\beta}$ such that xb = yc. Now, for a, d ∈ S, and since S is right reversible, there exists m, n in S such that md = na. This implies that $c^{-1}d\rho a^{-1}b$. Thus $a^{-1}b\rho c^{-1}d$ implies $c^{-1}d\rho a^{-1}b$. Therefore, ρ is symmetric. To prove transitivity, we assume that for $a^{-1}b$, $c^{-1}d$, $p^{-1}q \in Q$, let $a^{-1}b\rho c^{-1}d$ and $c^{-1}d\rho p^{-1}q$. This implies that there exists x, y in S such that xb = yc and m, n in $S_{\beta\gamma}$ such that md = np. Note that ρ, q are in $S_\gamma$. So,

xb = yc ............................ (1)

md = np ............................ (2)

Since, d, c are in $S_\beta$ and $S_\beta$ is right reversible, we have, td = sc, where t, s are in $S_\beta$. If u $S_\alpha$, then, utd = usc. But ut, us $\in S_{\alpha\beta}$. Let k = ut, v = us, then, kd = vc. Therefore, from (1), there exists $\ell$, r in $S_{\alpha\beta}$ such that

$$rb = \ell d$$

If g is in $S_\gamma$, then,

$$grb = g\ell d \text{ ........................ (3)}$$

and from (2)

$$gmd = gnp \text{ ...................... (4)}$$

Multiplying (3) by m and (4) by $\ell$, we have,

$$mgrb = mg\ell d \text{ .................. (5)}$$

$$\ell gmd = \ell gnp \text{ .................. (6)}$$

So,

$$sb = hd \text{ .......................... (7)}$$

and

$$hd = wp \text{ ......................... (8)}$$

implies

$$sb = wp.$$

This implies that $a^{-1}b\rho p^{-1}q$. Thus, ρ is an equivalence relation. Let

a, b $\in S_\alpha$, c, d $\in S_\beta$ and m, n ∈ S. We show that for $a^{-1}b$, $c^{-1}d$, $m^{-1}n \in Q$.

Suppose

$$a^{-1}b\rho^{-1}d \text{ ............................ (9)}$$

then by definition of ρ, there exists x, y in $S_{\alpha\beta}$ such that

$$xb = yc.$$

Also, since ρ is reflexive, we have

$$m^{-1}n\rho m^{-1}n \text{ .................... (10)}$$

That is, there exists s, t in $S_\gamma$ such that

$$sn = tm.$$

Using (9) and (10), we have,

$$m^{-1}n \bullet a^{-1}b\rho m^{-1}n \bullet a^{-1}b$$

if and only if there exists h, k in $S_{\alpha\beta\gamma}$ such that hvb = kqm, where v is in $S_{\gamma\alpha}$ and q is in $S_{\gamma\beta}$.

In [3], we note the following, that semilattice of right reversible semigroup is again right reversible semigroup. Thus for $S = \bigcup_{\alpha \in Y} S_\alpha$ where each $S_\alpha$, α ∈ Y is right reversible, S is right reversible. If a, b $\in S_\alpha$, c, d $\in S_\beta$ and m, n $\in S_\gamma$, then a, b, c, d, m, n ∈ S. Let b, m ∈ S, then by right reversibility of S, we have, h', k' ∈ S, such that H'b, = k'm

Suppose that k', h ∈ S and for the equation

hvb = kqm

Note that

$hv \in S_{\alpha\beta\gamma\gamma\alpha} = S_{\alpha\beta\gamma\alpha} = S_{\alpha\beta\gamma}\alpha$

and

$kq \in S_{\alpha\beta\gamma\gamma\alpha} = S_{\alpha\beta\gamma\gamma\alpha} = S_{\alpha\beta\gamma}$.

Therefore, if hv = h' and kq = k', then we have the required relation. Thus, ρ is congruence.

## 5.0 Construction of Normal Subgroup

Let us now use our congruence to construct a normal sub-semigroup of Q. Let us define the set $N_\rho$ as follows:

$N_\rho = \{a^{-1}b: a^{-1}bpe_\alpha\}$

Note that $a^{-1}b \in Q$ and $Q = \bigcup_{\alpha\varepsilon Y} G_\alpha$, where each $G_\alpha$ is the H-class and is a group.

Thus,

$N_\rho = \{a^{-1}b \in Q: xb = ye_\alpha;$ where $x, y \in S_{\alpha\beta}\}$. We now show that $N_\rho$ is a normal subgroup of Q. We shall maintain the above notations except otherwise stated.

Suppose $a^{-1}b, c^{-1}d \in N$, where a, b ∈ S, c, d ∈ $S_\beta$, then by definition, we have,

$a^{-1}bpe_\alpha$

and

$c^{-1}d\rho e_\beta$.

Here $e_\alpha \in G_\alpha$, $e_\beta \in G_\beta$. So, combining the two equivalent relations, we have

$(a^{-1}b)(c^{-1}d)\,\rho e_\alpha e_\beta$

That is,

$(xa)^{-1}yd\rho e_{\alpha\beta}$

where x, y ∈ $S_{\alpha\beta}$. Since the product $(xa)^{-1}yd$ is in $G_{\alpha\beta}$ and $e_{\alpha\beta} \in G_{\alpha\beta}$, then $(a^{-1}b)$ $(c^{-1}d) \in N_\rho$. Therefore, $N_\rho$ is a closed subset of the semigroup Q. We remark here that, if $a^{-1}b \in Q$, then $b^{-1}a \in Q$. Now, let $a^{-1}b\rho e_\alpha$, then, $(a^{-1}b)^{-1}\rho e_\alpha^{-1}$, that is, $b^{-1}a\rho e_\alpha^{-1}$ in $N_\rho$. Therefore, $(a^{-1}b)^{-1} = b^{-1}a \in N_\rho$. So $N_\rho$ is a subgroup of Q. Now, let $a^{-1}b \in N_\rho$ and $m^{-1}n \in Q$. So by definition of elements of $N_\rho$, we have

$a^{-1}b\rho e_\alpha,\ m^{-1}n\rho e_\beta$

implies

$(m^{-1}n)^{-1}pe^{-1}$,

that is

$(m^{-1}n)^{-1}\rho e_\beta$ or $(n^{-1}m)$ $\rho e_\beta$

Then, combining the equivalent relation, we have

$(m^{-1}n)\,a^{-1}bm^{-1}n\rho e_{\alpha\beta}$ and $a^{-1}b \in Q$

That is,

$(m^{-1}n)^{-1}a^{-1}bm^{-1}n \in N_\rho$

Thus, $N_\rho$ is a normal subgroup of Q.

### 5.1 Remark

The kernel of p is actually the normal subgroup.

### 5.2 Theorem

*Let Q be a left regular band of groups of left quotient, and $N_\rho$ is a normal subgroup of Q.*

*Then, $Q/N_{\rho}$ is also a left regular band of groups of left quotient.*

Proof

Let Q be a left regular band of groups of left quotient of the semigroup S. Our duty is to show that $S/N_{\rho}$ is a left order in $Q/N_{\rho}$. That is, we show that $S_{\alpha}/N_{\rho}$ is a semilattice of right reversible cancellative semigroups $S/N_{\rho}$. Let

$$Q/N_{\rho} = \cup_{\alpha \in Y}\left( G_{\alpha}/N_{\rho_{\alpha}} \right)$$

and

$$S/N_{\rho} = \cup_{\alpha \in Y}\left( S_{\alpha}/N_{\rho_{\alpha}} \right)$$

where

$$S_{\alpha}/N_{\rho\alpha} = S/N_{\rho} \cap G_{\alpha}/N_{\rho\alpha}$$

Any element of S/N is expressed in the form $a_{\alpha}N_{\rho\alpha}$. Now, for $a_{\alpha} \in S_{\alpha}$, $a_{\alpha}N_{\rho\alpha} \in S_{\alpha}/N_{\rho\alpha}$.

So

$$S_{\alpha}/N_{\rho\alpha} \neq \emptyset.$$

Let $a_{\alpha}^{-1}b_{\alpha} \in G_{\alpha}$, where $a_{\alpha} \in S_{\alpha}$, $b \in S_{\alpha}$, then

$$a_{\alpha}^{-1}b_{\alpha}N\rho_{\alpha}, \in G_{\alpha}/N_{\rho\alpha}$$

Note that $G_{\alpha}/N_{\rho}$ is a group and the identity of $G_{\alpha}/N_{\rho\alpha}$ is of the form $e_{\alpha}N_{\rho\alpha}$. So

$$c_{\alpha}^{-1}b_{\alpha}N_{\rho\alpha} = a_{\alpha}^{-1}b_{\alpha}e_{\alpha}N_{\rho\alpha}$$

Since every element of Q can be written in the form $a^{-1}b$, where $a, b \in S$, then every element of $Q/N_{\rho}$ can be written in the form $a^{-1}bN_{\rho}$. S is right reversible and cancellative. Thus $S_{\alpha}/N_{\rho\alpha}$ is also right reversible and cancellative. We recall that S is a semilattice Y of the semigroups $S_{\alpha}$, c $\in Y$. Therefore, $S/N_{\rho}$ is also a semilattice Y of the semigroups $S_{\alpha}/N_{\rho\alpha}$. Thus, $Q/N_{\rho}$ is a Clifford semigroup of left quotients of the semigroup $S_{\alpha}/N_{\rho\alpha}$.

**5.3 Theorem**

*Let Q be a left regular band of groups of left quotient. If $\rho$ is congruence on Q then $S/\rho$ is a left order in $Q/\rho$.*

Proof

To do this, we have to first show that $(a\rho)^{-1}$ is the inverse of $a\rho$ in a subgroup of $Q/\rho$.

In this case, we note that

$$(a\rho)^{-1} = a^{-1}\rho$$

Also, from [2] congruence on S is also congruence on Q. That is $\rho$ is congruence on S.

So,

$$(a\rho)^{-1}a\rho(a\rho)^{-1} = a^{-1}\rho a\rho a^{-1}\rho = (a^{-1}aa^{-1})\rho = (a\rho)^{-1}$$

and

$$(a\rho)(a\rho)^{-1}a\rho = (aa^{-1}a)\rho = a\rho$$

Now, S being a left order in Q implies that for

$x \in Q$, $x = a^{-1}b$, $a, b \in S$

and

$$a^{-1} \in G \leq Q$$

So,

$$x\rho = (a^{-1}b)\rho = a^{-1}\rho b\rho = (a\rho)^{-1}b\rho \in Q/\rho$$

where $a\rho, b\rho \in S/\rho$, $(a\rho)^{-1} \in G/\rho \leq Q/\rho$.

Thus, every element of $Q/\rho$ can be written in the form

$$a^{-1}\rho b\rho\ (a^{-1}b)\rho$$

ISSN 2229-5518

Since S is a left order in Q, every square-cancellable element of S is in a subgroup of Q. That is, if a is square cancellable in S, that is

$a^2x = a^2y$ implies that ax = ay, for x, y ∈ S

and

$xa^2 = ya^2$ implies that xa = ya, for x, y ∈ S

Then,

$a \in G \leq Q$.

Now, let

$(a\rho)^2(x\rho) = (a\rho)^2(y\rho)$, for aρ , aρ , yρ ∈ S/ρ

That is,

$(a^2\rho)(x\rho)\ (a^2\rho)\ (y\rho)$

That is,

$(a^2x)\rho = (a^2y)\rho$

That is,

(ax)ρ = (ay)ρ, since a is square cancellable.

That is,

(aρ) (xρ) = (aρ) (yρ)

Also, if

$(x\rho)(a\rho)^2\ (y\rho)(a\rho)^2$

Then,

(xρ)(aρ) = (yρ)(aρ)

Thus, aρ is square cancellable in S/ρ. By definition of square cancellability $a\rho\mathcal{H}'^*a^2\rho$ in S/ρ. We also note that if $a\rho\mathcal{H}^*a^2\rho$ in S/ρ, then $a\rho\mathcal{H}a^2\rho$ in Q/ρ. Since aρ is in a subgroup $G/\rho \leq Q/\rho$ if and only if $a\rho\mathcal{H}a^2\rho$ in Q/ρ, then every square-cancellable element of S/ρ is in a subgroup of Q/ρ. Thus S/ρ is a left order in Q/ρ.

**5.4 Note**

Let $a^{-1}b \in Q$, then $a^{-1}b \bullet a^{-1}b = (xa)^{-1}yb$, where a ∈, $S_\alpha$, b ∈ $S_\beta$ and x, y ∈ $S_{\alpha\beta}$. Now, if x = y, then,

$a^{-1}b^{-1}a^{-1}b = (xa)^{-1}xb = a^{-1}x^{-1}xb = a^{-1}b$.

That is, $a^{-1}b$ is an idempotent in Q if x = y. Taking this into consideration, we note that elements of the form $a^{-1}a$, $b^{-1}b$ are all idempotents in Q if the condition above holds. Since generally, $a^{-1}a$ is an idempotent in Q, then,

$((a\rho^{-1}a\rho)^2 = ((a^{-1}\rho)(a\rho))^2 = ((a^{-1}a)\rho)^2 = (a^{-1}a)^2\rho$

$= (a^{-1}a)\rho$

Thus, elements of $(a^{-1}a)\rho$ are the idempotents in Q/ρ.

**5.5 Definition**

The set of idempotents of Q/ρ is called the kernel of Q/ρ.

**5.6 Lemma**

The kernel of Q/ρ is a sub-semigroup of Q/ρ.

Proof

Let $a^{-1}a\rho$, $b^{-1}b\rho \in Q/\rho$ then,

$(a^{-1}a\rho)\ (b^{-1}b\rho) = (a^{-1}a)\ (b^{-1}b)\rho = ((xa^{-1}yb)\rho$, where x, y ∈ S. So, $[(xa)^{-1}yb]\rho \in Q/\rho$. Therefore, Q/ρ is closed.

Let $a^{-1}a\rho$, $b^{-1}b\rho$, $m^{-1}m\rho \in Q/\rho$ then,

$(a^{-1}a\rho b^{-1}b\rho)\ m^{-1}m\rho = [(a^{-1}ab^{-1}b)\rho]\ m^{-1}m\rho$

$= [(a^{-1}ab^{-1}b)\ m^{-1}m]\rho = [a^{-1}a(b^{-1}b\ m^{-1}m)]\rho$

$= [(a^{-1}a)\rho(b^{-1}b\ m^{-1}m)\rho]$

$= (a^{-1}a)\rho[(b^{-1}b)\rho(m^{-1}m)\rho]$

Thus, the kernel is a sub-semigroup of Q/ρ.

**5.7 Remark**

By definition, two idempotents are either equal or different. If a ∈ $S_\alpha$ and $a^{-1}a$ is an idempotent in Q, then $a^{-1}a\rho$ is an idempotent in Q/ρ. Considering the natural mapping of Q onto Q/ρ, we note that idempotent in Q is mapped into idempotent in Q/ρ.

**5.8 Notation**

We shall denote the kernel of Q/ρ by $\{N_{\rho_\alpha} : \alpha \in Y\}$. Viewed as elements of Q, each $N_{\rho_\alpha}$ contains an idempotent. [3] had proved that $N_{\rho_\alpha}$ is a normal subgroup. So idempotents in Q are contained in some element of A. Elements of $N_{\rho_\alpha}$ are of the form $a^{-1}a\rho$, where a ∈ $S_\alpha$.



**5.9 Lemma**

*Let $x^{-1}y \in Q$, then $(x^{-1}yf)^{-1}N_{\rho_\alpha} x^{-1}y \subseteq N_{\rho_\beta}$ for some $\beta$ in Y.*

Proof

We note that $(x^{-1}y)^{-1} = y^{-1}x$, and that $x^{-1} N_{\rho\alpha} x$ is an idempotent, since every element of $N_{\rho\alpha}$ is an idempotent. So,

$(x^{-1}y)^{-1} N_{\rho\alpha} x^{-1}y = y^{-1}xN_{\rho\alpha} x^{-1}y \subseteq N_{\rho_\beta}$, for some β ∈ Y, since

$y^{-1}x N_{\rho\alpha} x^{-1}y$ is an idempotent.

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