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Connecting sets of formalized operators

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Abstract - In this report is examined the generalization of the principles, that control the formalization of a single qubit operator. The set of the identical operators is used in order to be captured the types of operators, occurring at the application of negating operators in the n qubit space.

Index Terms— boolen function, circuit, composition, encoding, gate, phase, quantum.

1 Introduction

The formalization of single qubit operators aims to present all single bit operators as linear combinations of the identity and negation sets. Both classes are analogous to the classical operators and therefore the formalization of single qubit operators with these classes, acting as basis operators, provides means for operation with primitive operators, grounded in the classical concepts for computations. It has been proven that the space is enough to capture BQP, the class of tasks, efficiently solvable by quantum computations [1, 2]. In previous publications of the author [6, 7, 8] were defined several sets of operators on the n qubit, that generalize certain classical characteristics: identity and logical negation. Moreover, some ways were explored in which can be constructed operators as linear combinations of elements from those sets. Such combinations capture the partial application of an operator together with another operator, that in a broader sense is its logical negation.

2 Range of the *n* qubit operators

First, let's examine again the basic sets.

1. Ext_n and $N\mathrm{ext}_n$ are the operators, that generalize the identity and the negation respectively.
2. Int_n is the subset of Ext_n, that identifies the operators both probabilistically and physically.
3. $\mathrm{O}rtho_n$ is the set of operators, that connects states to an orthogonal state.
4. $Nint_n$ is the intersection point of $\mathrm{O}rtho_n$ and $N\mathrm{ext}_n$, and as such is the set of operators, that probabilistically and physically negate a given state.
5. Υ_n is the set of operators, that generally connect states to a different state, in other words, they obviously perform a "not" operation.

Until now was added a generalization, that encapsulates Υ_n - the "not" operators - and Ext_n - the operators for identity. Definition 1 defines the set of operators, that permute the basis set $\{|0\rangle, |1\rangle, \dots, |2^n - 1\rangle\}$, while at the same time introduce a possible phase change. Theorem 1 gives a decomposition for that set, that reflects that of Υ_n, given in Theorem 1.2.

Definition 1 *The set ρ_n includes all n qubit operators, for which $b, c \in \mathbb{B}^n$ and $V \in \rho_n$.*
$V|b\rangle = \pm|c\rangle$
Operators in ρ_n can be presented by 2^nx2^n signed permutation matrices. For each $V \in \rho_n$ there exists a permutation σ of $\{0, 1, \dots, 2^n - 1\}$, such that $V_{\sigma(i)i} = \pm 1$.

In contrast to the set Υ_n, ρ_n allows a state to be connected to itself. For some subsets of $\{0, 1, \dots, 2^n - 1\}$, *a* ρ_n operator may act as an Ext_n operator, while for the remaining subsets acts as an Υ_n operator. The operators in ρ_n are decomposed in respect of the phase encoding and permutations matrices in the same way as the Υ_n operators.

Theorem 1 *If $V \in \rho_n$ is a signed permutation operator such that for all basic states $|x\rangle$, V encodes phase function f on the permutation σ as follows,*
$V|x\rangle = (-1)^{f(x)}|\sigma(x)\rangle$
(25)

Then there exists the permutation matrix B and the operators $F, G \in \mathrm{Ext}_n$, that encode σ and the boolean functions f and g with $g = f^\circ\sigma^{-1}$, respectively, such that $V = B^\circ ID_f = ID_g{}^\circ B$. More over, $f = ID_f$ and $G = ID_g$.

Proof. If V is given, the operators F, G and B can be constructed as follows
$F_{ij} = (-1)^{f(i)}\delta_{ij}$
$C_{ij} = (-1)^{g(i)}\delta_{ij} = (-1)^{f(\sigma^{-1}(i))}\delta_{ij}$

where $(-1)^{f(i)}$ is the non-zero value in column i of V and $(-1)^{f(\sigma^{-1}(i))}$ is the non-zero value in column i of $V^\dagger$ or an alternate row i of V. For operator B
$B_{ij} = |V_{ij}|$

for all $i, j \in \{0, 1, \dots, 2^n\}$. It then follows that for each $x \in \mathbb{B}^n$

$$\begin{aligned} BF|x\rangle &= (-1)^{f(x)}B|x\rangle \\ &= (-1)^{f(x)}|\sigma(x)\rangle \\ GB|x\rangle &= C|\sigma(x)\rangle \\ &= (-1)^{f\left(\sigma^{-1}\sigma(x)\right)}|\sigma(x)\rangle \\ &= (-1)^{f(x)}|\sigma(x)\rangle \end{aligned}$$

Theorem 2 establishes the basic relations between the operators sets $\rho_n, \Upsilon_n, \mathrm{O}rtho_n, N\mathrm{ext}_n$ and Ext_n.

Theorem 2 *For n > 1*
1. $\Upsilon_n \subset \rho_n$
2. $\mathrm{Ext}_n \subset \rho_n$

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$\mathrm{O}rtho_n \subset \Upsilon_n$

4. $Next_n \subset \Upsilon_n$

Proof.

1. $\Upsilon_n \subset \rho_n$

From the definition of ρ_n and Υ_n is visible, that both sets contain signed permutation matrices and $\Upsilon_n \subseteq \rho_n$. However, ρ_n allows operators, that connect one or more states with themselves. Such operators do not exist in Υ_n and therefore $\Upsilon_n \subset \rho_n$.

2. $\mathrm{Ext}_n \subset \rho_n$

The set Ext_n can also be defined as signed permutation matrices, that encode the permutation σ such that $\sigma(i) = i$. From this follows that $\mathrm{Ext}_n \subseteq \rho_n$. Given that $\Upsilon_n \subset \rho_n$ and $\Upsilon_n \cap \mathrm{Ext}_n = \emptyset$, it follows that $\mathrm{Ext}_n \subset \rho_n$.

$\mathrm{O}rtho_n \subset \Upsilon_n$ и $Next_n \subset \Upsilon_n$

From Theorem 4.2.2, that characterizes the matrix presentation of an $\mathrm{O}rtho_n$ operator, it follows that each $A \in \mathrm{O}rtho_n$ is a signed permutation matrix, in which no state is linked to itself and therefore $\mathrm{O}rtho_n \subseteq \Upsilon_n$. Additionally, Theorem 4.2.1 clearly places any $N\mathrm{ext}_n$ operator in Υ_n and therefore $N\mathrm{ext}_n \subseteq \Upsilon_n$. Now let's consider the operator $N \otimes N$.

$$N \otimes N = \begin{pmatrix} 0 & -N \\ N & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad (26)$$

It is known that $N \otimes N \in \Upsilon_n$. If it is in $\mathrm{O}rtho_n$, then for any two-qubit state $|\phi\rangle$, $\langle\phi|N \otimes N|\phi\rangle = 0$. However, $\langle\phi|N \otimes N|\phi\rangle = 2(\phi_0\phi_3 - \phi_1\phi_2)$, which is 0 only when $\phi_0\phi_3 - \phi_1\phi_2 = 0$. $\mathrm{O}rtho_n \neq \Upsilon_n$It follows then that $\mathrm{O}rtho \subset \Upsilon$ and so by corollary 4.2.3 it is clear that there exists operators in $\mathrm{O}rtho_n$, but not in $Next_n$, and therefore $N\mathrm{ext}_n \subset \Upsilon_n$. In the hierarchy are placed the elementary indexed operator and the two-qubit controlled operator. The set $U_{\mathcal{B}^1|n}$, as defined in Definition 1.1, contains all two-qubit controlled operators. Definition 2 introduces the sets, containing the indexed basis operators.

Definition 2.

$\mathrm{Ext}_{1|n} = \{A_{[t]} | 0 \le t < n, A \in \mathrm{Ext}_1\}$

$Int_{1|n} = \{A_{[t]} | 0 \le t < n, A \in Int_1\}$

$Next_{1|n} = \{A_{[t]} | 0 \le t < n, A \in Next_1\}$

$Nint_{1|n} = \{A_{[t]} | 0 \le t < n, A \in Nint_1\}$

From these definitions may be established the ratio between the indexed form of the single qubit basic gates and the broader space of the *n* qubit basis operators. Theorem 3 creates these relationships for the identity operators and Theorem 4 makes the same for the negation operators.

Theorem 3 *For n* > 1

1. $Int_{1|n} = Int_n$

2. $\mathrm{Ext}_{1|n} \subset \mathrm{Ext}_n$

Proof. From equation 1, it follows that for each $t, I_{[t]} = I_n$ and $-I_{[t]} = -I_n$. Thus $nt_{1|n} = Int_n$. From 1 it is seen that for each $A \in \mathrm{Ext}_1, A_{[t]} \subseteq \mathrm{Ext}_n$. However, for all $i \in [0, 2^n), \langle i|A_{[t]}|i\rangle = (-1)^{E(p_A)0(x_t)}$. If it is given, that Ext_n allows random phase assignments, can be constructed $B \in \mathrm{Ext}_n$ such that for *i*, *j* with $i_t = j_t$, but $i \neq j$ $\langle i|B|i\rangle \neq \langle j|B|j\rangle$.

Theorem 4 *For n* > 1

1. $N\mathrm{ext}_{1|n} \not\subseteq Next_n$

2. $N\mathrm{ext}_{1|n} \subset \Upsilon_n$

$Nint_{1|n} \subset \mathrm{O}rtho_n$

Proof. From equation 1 follows that for each $A \in N\mathrm{ext}_1, A_{[t]} \notin N\mathrm{ext}_1$, since A connects $|x\rangle$ with $\pm|x \oplus 2^t\rangle$, and not $\pm|x\rangle$. However, from the same observation it is seen that $N\mathrm{ext}_{1|n} \subset \Upsilon_n$, as no state will be connected to itself. Furthermore, there exists a non-empty subset of Υ_n, namely $N\mathrm{ext}_n$, in which can not be found $N\mathrm{ext}_{1|n}$ operator. For $A \in Nint_{1|n}$ it is known from equation 2, that all non-zero entries are found on $A_{i,i\oplus 2^t}$. Since $A_{i,j} + A_{j,i} = \langle j|A|i\rangle + \langle i|A|j\rangle$ follows that, for all i and j such that $j \neq i\oplus 2^t$, $A_{i,j} + A_{j,i} = 0$. When $j = i\oplus 2^t$, it is known that $\langle j|A|i\rangle + \langle i|A|j\rangle = (-1)^{\mathcal{E}(p_A)_1(x_t)} + (-1)^{\mathcal{E}(p_A)_1(\overline{x_t})}$

If it is given that $\mathcal{E}_{(p_A)_1} \in BAL$, then $\langle j|A|i\rangle + \langle i|A|j\rangle = 0$ and $A \in \mathrm{O}rtho_n$.

After the indexed forms of the single-qubit basis operators, placed in the hierarchy of the *n* qubit basis operators, were examined, now the attention can be focused at the controlled operators. Theorem 5 establishes the location of $U_{\mathcal{B}^1|n}$ within the hierarchy, while Theorem 6 establishes the relationships between the controlled and indexed operators.

Theorem 5 *For n* > 1

1. $U_{\mathcal{B}^1|n} \subset \rho_n$
2. $U_{ID|n} \not\subseteq \Upsilon_n \cup \mathrm{Ext}_n$
3. $U_{NOT|n} \not\subseteq \Upsilon_n \cup \mathrm{Ext}_n$
4. $U_{ZERO|n} \subset \mathrm{Ext}_n$
5. $U_{ONE|n} \subset \Upsilon_n$

Proof. It is clear from their definitions, that $U_{\mathcal{B}^1|n} \subseteq \rho_n$. Furthermore, from the characterization of the matrices of $U_{\mathcal{B}^1|n}$ is seen, there does not exist an operator in $U_{\mathcal{B}^1|n}$, that i also in $N\mathrm{ext}_n$, as at most one bit can be reversed by an operator in $U_{\mathcal{B}^1|n}$, and $N\mathrm{ext}_n$ requires all bits to be reversed. From there $U_{\mathcal{B}^1|n} \subset \rho_n$. From the characterization of the operators in $U_{ID|n}$ and $U_{NOT|n}$, given in Definition 2.1 is visible, that the operators from both sets connects some basic states with themselves and others - with the state with reversed bit *t*. The matrix for such an operator, as shown in corollary 2.2 is neither in Ext_n, nor in Υ_n. Thus, $U_{ID|n} \not\subseteq \Upsilon_n \cup \mathrm{Ext}_n$ and $U_{NOT|n} \not\subseteq \Upsilon_n \cup \mathrm{Ext}_n$.

From corollary 2.2 and Theorem 2.4.1.2 it is visible, that $U_{ZERO|n} \subseteq \mathrm{Ext}_n$. However, for each allowable control bit *c*, can be constructed $A \in \mathrm{Ext}_n$ such that for $f, g \in \mathcal{B}^1$ $f \neq g$ and

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$i \in [0, 2^n)$ $\langle i|A|i\rangle = (-1)^{f(x_c)}$ and all other entries on the diagonal are equal to $(-1)^{g(x_c)}$. From corollary 2.2 it is clear that such an operator is not in $U_{ZERO|n}$.
From corollary 2.2 and Definition 1.1 it is visible that $U_{ONE|n} \subseteq \Upsilon_n$, as the $U_{ONE|n}$ operators always connect $|x\rangle$ with $\pm|x \oplus 2^t\rangle$. However, it is also clear that for $n > 1$ none $U_{ONE|n}$ operator does not connect $|x\rangle$ with $|\bar{x}\rangle$ and therefore there exists a non-empty subset of Υ_n, which does not cross with $U_{ONE|n}$.

Theorem 6. *For* $n > 1$
1. $Next_{1|n} \subset U_{ONE|n}$
2. $Ext_{1|n} \subset U_{ZERO|n}$

Proof. If operator $V = CU_{[c][t]}(A, A) \in U_{ZERO|n}$ is such that $A \in \mathrm{Ext}_1$. It can easily be checked that $V = A_{[t]}$ and therefore $\mathrm{Ext}_{1|n} \subseteq U_{ZERO|n}$. If it is selected $B \in \mathrm{Ext}_1$ such that $A \neq B$ and $V = CU_{[c][t]}(A, B) \in U_{ZERO|n}$. In such case V is equivalent to an indexed operator, as it applies two different phase changes operators to two different subspaces of the n qubit space, as shown in Definition 2.1. Therefore $\mathrm{Ext}_{1|n} \subset U_{ZERO|n}$. The same argument can be used relative to $U_{ONE|n}$, only with $A, B \in N\mathrm{ext}_1$, in order to be shown that $N\mathrm{ext}_{1|n} \subset U_{ONE|n}$. Theorem 6 establishes the last relationships, that are necessary to construct a hierarchy of elementary operators.

3 Conclusion

This report began with a simple extension of the formalized operator, that acts on a single qubit in a single qubit space. The effect of such operator is given from equation 1. At the generalization of the principles, that manage the formalization of the single qubit operator, the set Υ_n of the operators was established in Definition 1, in order to capture the types of operators, occurring at the application of $N\mathrm{ext}_1$ operators in the n qubit space. This in turn admitted Theorem 3, that characterizes the extension of the single qubit operators to an n qubit space in a way similar to the single qubit operators in a single qubit space and provided an unified view of the operators, which act on a single qubit.

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