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Decomposition of Pregroups

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Abstract— In section one, we introduced the main concept and definition which we needed in later sections. In section two, we proved that some axioms are equivalent to the other ones. Section three contains the main features of work.

In section three of this paper we proved that any pregroup satisfying P_6 can be expressed as a product of factors which are also pregroups satisfying P_6. We also proved that the universal group of a pregroup satisfying P_6 is the free product of the universal groups of the factors of P amalgamating the core part P_o.

Index Terms— Amalgamation, Core, Decomposition, Factors, Free Prducts, Factors, Length of a word, Pregroup, Reduced words, Universal Group.

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1 Introduction

Stallings [5] in 1971 introduced the concept of a pregroup. Subsequent work has been dose by Nesayef [3], 1983, Chiswell [1], 19jm bn87 and many others.

Five axioms were originally introduced by Stallings [5], namely P_1, P_2, P_3, P_4, and P_5. We proved that P_3 is a consequence of the other axioms and we proved that P_6 which was introduced by Nesayef [3] is stronger than P_5.

Stallings [5] introduced the fallowing construction of a pregroup.

Definition 1.1: A pregroup is a set P containing an element called the identity element of P, denoted by 1, a subset D of $P \times P$ and a mapping $D \rightarrow P$, when $(x, y) \rightarrow xy$ together with a map $i : P \rightarrow P$ when $i(x) = x^{-1}$, satisfying the following axioms .
(we say that x y is defined if $(x, y) \in D$, i.e. $xy \in P$).

P_1 . For all $x \in P$, 1x and x1 are defined and $1x = x1 = x$.
P_2 . For all $x \in P$, $x^{-1}x = x^{-1}x = 1$
P_3 . For all $x, y \in P$ if x y is defined , then $y^{-1}x^{-1}$ is defined and $(xy)^{-1} = y^{-1}x^{-1}$.
P_4 . Suppose that $x, y, z \in P$. If x y and y z are defined, then x (y z) is defined, is which case $x(yz) = (xy)z$.
P_5 . If $w, x, y, z \in P$, and if w x, x y, y z, are all defined them either $w(xy)$ or $(xy)z$ is defined.

Proposition 1.2 : Let P be a pregroup and $a, x \in P$. If a x is defined, them $a^{-1}(ax)$ is defined and $a^{-1}(ax) = x$.

Proof : By P_2, we have $a^{-1}a$ is defined and equals 1.

Thus by P_4 and P_1, we have $a^{-1}(ax)$ is defined and $a^{-1}(ax) = (a^{-1}a)x = x$.

The following propositions prove that P_3 is a consequence of the other axioms.

Proposition 1.3 : Let P be a pregroup and $x, y, \in P$. If x y is defined them $y^{-1}x^{-1}$ is defined and $(xy)^{-1} = y^{-1}x^{-1}$

Proof : Suppose x y is defined. Then $xy \in P$ and $(xy)^{-1} \in P$

Consider : $x^{-1}, xy, (xy)^{-1}$:

$x^{-1}(xy)$ and $(xy)(xy)^{-1}$ are defined .

Since $x^{-1}[(xy)(xy)^{-1}]$ is defined and equals to x^{-1} then by P_4, we have :

$[x^{-1}(xy)](xy)^{-1}$ is also defined and equals to

$x^{-1}[(xy)(xy)^{-1}] = x^{-1}$

By P_4 again : $y(xy)^{-1} = x^{-1}$

Now consider : $y^{-1}, y, (xy)^{-1}$:

$y^{-1}y$ and $y(xy)$ are both defined .

Since $[y^{-1}y](xy)^{-1}$ is defined and $= (xy)^{-1}$.

Then by P_4 : $y^{-1}[y(xy)^{-1}]$ is also defined and $= (xy)^{-1}$.

Definition 1.4 : Let P be pregroup. A **word** in P is an n-tuple: $(x_1 \ldots x_4)$ of elements of P , for some $n \geq 1$. n is called the

length of the word .

Definition 1.5 : A word ($x_1 \ldots x_n$) is said to be reduced if $x_i\, x_{i+1}$ is not defined for any $1 \leq i \leq n-1$.

Let $P_0 = \{ x \in P : x\, y$ and $y\, x$ are defined for all $y \in P \}$. We call P_o the **core** of P .

Proposition 1.6 : P_o is a subgroup .

Proof : Suppose $x \in P_o$.

By the definition of P_o : x y , y x are defined for all $y \in P$ and by proposition 2: $y^{-1} x^{-1}$ and $y^{-1}x$ are both defined , so $x^{-1} \in P$

Suppose $x\, y \in P_o$. x y , y z and x (y z) are all defined for all $z \in P$.
By P_4 : (x y) z defined for all $z \in P_o$.

We now introduce an additional condition on a given pre-group P:
P_6 : Suppose (x y) is reduced. If x a and $a^{-1}y$ are both defined then $a \in P_0$.

It has been proved in [3] that p_6 is equivalent to :
$P_{6'}$: If (x , y) is reduced and (a x) y is defined for $a \in P_o$.

A further equivalence statement of P_6 is given by Hoare [2] as follows:

$P_{6''}$: If x y and y^{-1} z are defined and $y \in P_o$, then x z is defined.

Definition 1.7 : Let P be any pregroup. The **Universal group** U (P) is the set of all equivalence classes of reduced words.

2. Decomposition of Pregroups

Theorem 2.1: Any pregroup satisfying P_6 , can be expressed as a product of factors P_i , where each factor P_i is a pregroup satisfying P_6 .

To prove this theorem, we need the following:

Definition 2.2 :

Let P be pregroup satisfying P_6 and $P_o \neq 1$. Define a relation $\approx$ on $P \setminus P_o$ by:

$x \sim y$ if and only if $\exists\ a \in P_o$ such that x a y is defined .

Proposition 2.3 : The relation ~ is as equivalence relation .

Proof : This is reflexive for $x\, 1\, x^{-1} = 1 \in P$.

Symmetric : for if $x\, a\, y^{-1}$ is defined , these $[(x\, y)\, y^{-1}]^{-1} = y\, (x\, a)^{-1} = y\, a^{-1}\, x^{-1}$.

For transitivity, suppose x a y and $y\, b\, z^{-1}$ both defined. Since $y \in P_o$ then by P_6 , $x\, a\, b\, z^{-1}$ is defined .

i e $x \sim z$.

Therefore, "~" is an equivalence relation.

Definition 2.4 : Define a relative $\approx p \setminus p_0$ by :

$x \approx y$ if either $x \sim y$ or if $\exists z \in P \setminus P_o$ such that x ~ and y ~ z^{-1}.

Proposition 2.5 : The relation $\approx$ is an equivalent relative .

Proof : This is reflexive and symmetric .
For transitivity we assume that $x \approx y$. Then $\exists$ u and v such that $x \approx u$, $y \approx u^{-1} \approx v$ and $z \approx v^{-1}$.
Since $u^{-1} \approx v$, then $\exists\ a \in P_o$ such that v a u is defined .

By Proposition 2.3 v^{-1} (v a u) and (v a u) u^{-1} are defined and equal to au and va respectively.

So v^{-1} (v a u) u^{-1} is defined.
If $v\, a\, u \in P_o$, then $v^{-1} \sim u$ by definition of ~, $x \sim z$.
If $v\, a\, u \notin P_o$, then $v\, a\, u \sim u \sim x$ and $v^{-1} \sim u^{-1}\, a^{-1}\, v^{-1} \sim z$.
Hence $x \approx z$.
Therefore $\approx$ is an equivalence relation .

Proof of Theorem 1 :

Denote the class containing x_i under $\approx$ by $[x_i]$.
Let $P_i = P_o \cup \{ y \in [x_i]$ and $y^{-1} \in [x_i]$ and $y^{-1} \in [x_i] \}$.
For x_1 and x_2 in P_i , the product $x_1 x_2$ is defined in P_i if and only if $x_1 x_2$ is defined in P .

We now show that P_i is a pregroup satisfying P_6.
P_1 and P_2 are clear from the definition of P_i. So we need only consider P_4 and P_6 .

For P_4, let x , y and $z \in P_i$ and let x y and y z be defined in P_i . Suppose (x y) z is defined in P_i . Then x y , y z and (x y) z are defined in P.

By P_4 on P, we have x (y z) is defined in P hence it is defined in P_i.

To prove P_6 , suppose ($x_1.x_2$) is reduced in P_i , then ($x_1.x_2$) is reduced in P.

If x_1 and $a^{-1}x_2$ are both defined in P_i , then they are defined in P. By P_6 on P, we have $a \in P_o$.

Therefore P_i is a pregroup satisfying P_6 and denote $P = *_{P_o} P_i$.

3. The Universal Group :

Theorem 3.1: The universal group of a pregroup satisfying P_6 is isomorphic to the free product of the universal groups of the factors P_i of P , amalgamating P_o .

To prove this theorem, we need the following :

Definition 3.1: For x_1 and $x_2 \in P_i \setminus P_o$, we say that ($x_1 . x_2$) is **reduced** if $x_1 x_2$ is not defined in P_i.

Lemma 3.2: If $x_i \in P_i$, $x_j \in P_j$ for $i \neq j$, then (x_i , a x_j) is reduced for all $a \in P$.

Proof : If not then , there exists $a \in P_o$ such that x_i a x_i is defined, then $x_i x_j^{-1}$ is defined. So $x^{-1} \in P$, then $x_j \in P_i$ a contradiction .

Now let U (P_i) be the universal group of the pregroup P_i defined in stalling [4], further details with proofs are given in [4] .

Let U (P_i) be the universal group of the pregroup P_i and form $*P_o U(P_i)$.

Definition 3.3 : A sequence $x_1 . x_2 \ldots .x_n$ is reduced in $* P_o U(P_i)$, if $x_i x_j$ is not defined for $1 \leq i \leq n-1$, $2 \leq j \leq n$.

Indeed this expression is not unique, but we only need a reduced form of the elements of $*P_o U(P_i)$ in the theorem .

Proof of the Theorem :

Define the map $\phi: *P_o U(P_i) \rightarrow U(P)$ by :
$\phi(x_1. \ldots .x_n) = x_1x_2 \ldots x_n \in U(P)$, where $x_1 , x_2 , \ldots , x_n$ is in reduced form in $*P_o U(P_i)$

Suppose $\phi(x_1 . x_2 . \ldots . x_n) = x_1 x_2 \ldots x_n = 1$, for $n \geq 1$.
If $n = 1$, then $\phi(x_1) = x_1 = 1$
If $n > 1$, then there exists some i for which $x_i x_{i+1}$ is defined in P.

Let $x_i \in P_i$, then $x_{i+1} \in P_i$, moreover , $x_i x_{i+1}$ is reduced in P_i , a contradiction to reduced form in $*P_o U(P_i)$.
Hence ϕ is one- to - one.
Since P generate U (p) , then ϕ is onto . i.e. ϕ is an isomorphism .

Therefore U (P) is a free product of U (P_i) amalgamating P_o.

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