FRACTIONAL SECOND GRADE FLUID PERFORMING SINUSOIDAL MOTION IN A CIRCULAR CYLINDER

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Abstract— The purpose of this work is to obtain some new results for fractional second grade fluid (non-Newtonian) performing sinusoidal motion. The exact solution of the velocity field and associated shear stress corresponding to second grade fluid in an infinite cylinder are obtained by applying Laplace transform and Hankel transform. The solutions have been written in series form using generalized function G.,.,.(.,t) function and Bessel function. For $\alpha \to 0$ similar solutions for Newtonian fluid performing the same sinusoidal motion are obtained. Solutions for ordinary second grade fluid performing the same sinusoidal motion are recovered from fractional fluid (second grade) as a limiting case.

Index Terms— Exact solution, Fractional Second Grade fluid, Shear stress.

1 INTRODUCTION

The study of behavior of materials at rest or in motion which deforms without any limit under the influence of shearing forces is of considerable importance for researchers. Broad application of fluids in our daily life such as air we breathe, water, blood which runs through our body, applications in food industry, polymers chemical industry, drilling operation, and bio engineering makes this most fascinating field for researchers. Mass and heat transfer is of considerable importance in chemical engineering. Now a days in medical sciences, physiological fluid dynamics has become an important area of research. Fluids may be synthetic or natural. They are mixture of different stuffs e.g oils, red cells, water etc. These kind of fluids mostly have the viscosity, which has non-linearly variation with the deformation of fluids. The elasticity can be found through elongational effect and time dependent effects. In this case, fluids have been treated as viscoelastic. Fluid in which shear stress is not directly proportional to deformation rate are non-Newtonian fluids. One of the non-Newtonian fluid model which represent other rheological characteristics is differential type fluid (Rivilin-Erickson fluid) which mostly consists of liquid foams, polymeric fluids, slurries and food products and many substances which are capable of flowing, but which exhibit flow characteristics that cannot adequately describe by classical linear viscous fluid model.

A subclass that has gained a special attention is incompressible homogeneous second grade fluid. Equation of motion of the second grade fluid is third order partial differential equation. The second grade fluid equations are a model for viscoelastic fluid flow depending on two parameters, the elastic response α and viscosity v. Common examples of non-Newtonian viscoelastic second grade fluid are blood, fluids used in industrial fields, such as polymer solutions and certain kinds of oil. Galdi, Padula and Rajagopal studied the stability of flows of second grade fluids. Some recent attempts regarding exact analytical solution for the flow of the second grade fluid have been made by A. Mahmood [10], M. Jamil [7] M. Athar [1]. Some important studies of non-Newtonian fluids and oscillating boundary value problems in infinite cylinders are determined by A. Mahmood [9] who found exact solution of viscoelastic non-Newtonian (Second grade) fluid corresponding to longitudinal oscillatory flow. Whereas D. Vieru [16] found exact solution for the motion of Maxwell fluid due to longitudinal and torsional oscillations of an infinite cylinder by means of Laplace transform; while Cornia Fetecau [5] determined exact solution for oscillating Oldroyd-B fluid, between two infinite coaxial cylinders by using Laplace and Hankel transform. The governing equations for the motion of fractional fluid can be obtained by replacing inner time derivatives by the Riemann Liouville D_t^{β} in governing equations in ordinary fluid where D_t^{β} is defined in [3]. The idea of fractional derivative was introduced by Leibnitz in early 18th century to get answer to $\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}$ i.e. derivative of order $\frac{1}{2}$. Others who tried with the idea include L'Hospital, Euler, Lagrange, Riemann, Laplace, Liouville and Fourier. An excellent discussion of fractional differential equations and a good history of fractional calculus is given by K. S. Miller [12]. Carl F. Lorenzo [3] presented a very useful paper on fractional derivatives. Fractional derivatives are much flexible in describing viscoelastic behaviour of fluids [6,15]. Keeping in view the above

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discussion there is a need to solve velocity field and shear stress of generalized second grade fluid performing sinusoidal motion with different conditions.

2 Governing Equations

In this work we will consider the velocity V and the extra-stress S where

$$V = V(r,t) = \omega(r,t)e_\theta,\ S = S(r,t) \quad \ldots\ldots(1)$$

Where e_θ is the unit vector in the θ direction of the cylindrical coordinate system r, θ, z. Where at $t = 0$ we have $\omega(r,0) = 0$

Consider governing equations of ordinary second grade fluid

$$\tau(r,t) = (\mu + \alpha_1 \frac{\partial}{\partial t})\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t) \quad \ldots\ldots(2)$$

$$\frac{\partial \omega(r,t)}{\partial t} = (\vartheta + \alpha \frac{\partial}{\partial t})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t) \quad ..(3)$$

Where μ is dynamic viscosity of the fluid and $\alpha = \alpha/\rho$ is a material constant. $\theta = \mu/\rho$ is the kinematic viscosity of the fluid where ρ being its constant density and $\tau(r,t) = S_{r\theta}(r,t)$ is the shear stress.

The governing equations corresponding to an incompressible fractional second grade fluid are obtained from equation (2) and (3) by replacing inner derivative with respect to "t" by fractional derivative (fractional differential operator) D_t^β and $\beta > 0$ where

$$D_t^\beta f(t) = \left[\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^\beta}d\tau, \qquad 0 \le \beta < 1\right.$$
$$= \frac{d}{dt}f(t), \qquad \beta = 1 \quad \ldots\ldots(4)$$

Therefore, governing equatins for calculations in this paper are

$$\tau(r,t) = (\mu + \alpha_1 D_t^\beta)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t) \quad \ldots\ldots\ldots(5)$$

$$\frac{\partial \omega(r,t)}{\partial t} = (\vartheta + \alpha D_t^\beta)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t) \quad \ldots\ldots.(6)$$

3 Flow Through an Infinite Cylinder Having Shear on Boundary

Let us consider an incompressible fractional second grade fluid at rest, in an infinitely long cylinder of radius $R > 0$. At time $t = 0$ fluid is at rest and at time $t = 0^+$ cylinder begins to rotate and boundary of cylinder applies a sinusoidal shear stress on fluid. The fluid is gradually moved. Its velocity is of the form (1). The governing equations are given by (5) and (6). Appropriate initial and boundary conditions are

$$\omega(r,0) = 0 \qquad r\varepsilon(0,R] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(7)$$

$$\tau(R,t) = (\mu + \alpha_1 D_t^\beta)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t)\Big|_{r=R}$$
$$= \Omega R.Sin(\omega t) \quad with \quad t > 0$$

$$\Omega \text{ is constant}$$

3.1 Computation of the Velocity Field

Laplace transform of (6) and (7)

$$q\overline{\omega}(r,q) = (\vartheta + \alpha q^\beta)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\overline{\omega}(r,q) \quad \ldots\ldots(8)$$

$$\overline{\tau}(R,q) = (\mu + \alpha_1 q^\beta)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\overline{w}(r,q)\Big|_{r=R} = \frac{\Omega\omega.R}{(q^2+\omega^2)} \quad \ldots..(9)$$

Now we have to apply finite Hankel transformation to (8) and using (9) and result

$$\int_0^R r\left(\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\overline{\omega}(r,q)J_1(rr_n)dr =\right.$$
$$RJ_1(Rr_n)\left|\frac{\partial\overline{\omega}(r,q)}{\partial r} - \frac{1}{r}\overline{\omega}(r,q)\right|_0^R - r_n^2\overline{w}_H(r_n,q) \quad \ldots(10)$$

Where we denote Hankel transformation (finite) of $\overline{\omega}$(r,q) by [4]

$$\overline{w}_H(r_n,q) = \int_0^R r\overline{w}(r,q)J_1(rr_n)dr$$

Where $J_1(rr_n)$ represents Bessel function of first kind

We get finite Hankel transform of (8)

$$\overline{\omega}_H(r_n,q) = \left(\frac{RJ_1(Rr_n)\Omega\omega R}{(\mu r_n^2)(q^2+\omega^2)}\right) - \left(\frac{RJ_1(Rr_n)\Omega\omega Rq(1+q^{\beta-1}\alpha r_n^2)}{(\mu r_n^2)(q^2+\omega^2)(q+\vartheta r_n^2+\alpha q^\beta r_n^2)}\right)$$

We have

$$\overline{\omega}_{1H}(r_n,q) = \frac{1}{\mu r_n^2}\left(\frac{RJ_1(Rr_n)\Omega\omega R}{q^2+\omega^2}\right) \quad \ldots\ldots\ldots\ldots(11)$$

And

$$\overline{\omega}_{2H}(r_n,q) = -\left(\frac{RJ_1(Rr_n)\Omega\omega Rq(1+q^{\beta-1}\alpha r_n^2)}{\mu r_n^2((q^2+\omega^2)(q+\vartheta r_n^2+\alpha q^\beta r_n^2)}\right) \quad \ldots(12)$$

Applying Hankel Inverse transform to (11) and (12) and adding we get $\overline{\omega}$(r,q) where inverse Hankel transformation of

$\overline{\omega}_H(r_n,q)$ is defined by

$$\overline{\omega}(r,q)=\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1^2(Rr_n)}\overline{\omega}_H(r_n,q)$$

Where $J_1(Rr)=0$ at positive root r_n

$$\overline{\omega}_1(r,q)=\frac{\omega\Omega}{2R\mu(q^2+\omega^2)}.r^3 \qquad \text{......................(13)}$$

$$\overline{\omega}_2(r,q)=-2\sum_{n=1}^{\infty}\frac{J_1(rr_n)\Omega\omega}{\mu(r_n^2)J_1(Rr_n)}\left[\frac{q(1+q^{\beta-1}\alpha r_n^2)}{(q^2+\omega^2)(q+\vartheta r_n^2+\alpha q^{\beta}r_n^2)}\right] \qquad \text{..(14)}$$

Now apply Laplace inverse transform to (13) and (14) and adding we get velocity field

$$\omega(r,t)=\frac{\Omega r^3}{2\mu R}\sin(\omega t)-2\sum_{n=1}^{\infty}\frac{J_1(rr_n)\Omega}{\mu(r_n^2)J_1(Rr_n)}$$

$$\times\left[\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t\sin(\omega s)G_{1-\beta,1-\beta-\beta k,k+1}(-\alpha r_n^2,t-s)ds+\right.$$

$$\left.\alpha r_n^2\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t\sin(\omega s)G_{1-\beta,-\beta k,k+1}(-\alpha r_n^2,t-s)ds\right] \text{...(15)}$$

Where Generalized function $G_{a,b,c}(.,.)$ is defined by [3] equation (97) and (100)

3.2 Computation of Shear Stress

Applying Leplace transform to (5) we get

$$\overline{\tau}(r,q)=(\mu+\alpha_1q^{\beta})\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\overline{\omega}(r,q) \qquad \text{...........(16)}$$

Writing $\overline{\omega}_H(r_n,q)$ into equivalent form with

$$\overline{\omega}_{1H}(r_n,q)=\left(\frac{\Omega R^2\omega J_1(Rr_n)}{r_n^2}\right)\left(\frac{1}{(q^2+\omega^2)(\mu+\alpha_1q^{\beta})}\right) \qquad \text{...(17)}$$

and

$$\overline{\omega}_{2H}(r_n,q)=-\left(\frac{\Omega R^2\omega J_1(Rr_n)}{r_n^2}\right)\left(\frac{q}{(q^2+\omega^2)(\mu+\alpha_1q^{\beta})(q+\vartheta r_n^2+\alpha q^{\beta}r_n^2)}\right) \qquad \text{...(18)}$$

Applying inverse Hankel transform to (17) and (18) and adding substituting in (16)

$$\overline{\tau}(r,q)=\frac{\Omega\omega r^2}{R(q^2+\omega^2)}+2\sum_{n=1}^{\infty}\left(\frac{J_2(rr_n)\Omega}{(r_n)J_1(Rr_n)}\right)\left[\frac{q\omega}{(q^2+\omega^2)(q+\vartheta r_n^2+\alpha q^{\beta}r_n^2)}\right]$$

Apply Laplace inverse transformation

$$\tau(r,t)=\frac{\Omega r^2}{R}Sin(\omega t)+2\sum_{n=1}^{\infty}\frac{J_2(rr_n)\Omega}{(r_n)J_1(Rr_n)}$$

$$\times\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t\sin(\omega s)G_{1-\beta,1-\beta-\beta k,k+1}(-\alpha r_n^2,t-s)ds \qquad (19)$$

4. Limiting Cases

4.1 Newtonian Fluids

Applying $\alpha\to0$ and $\alpha_1\to0$ in (15) and (19)

$$\omega(r,t)=\frac{\Omega r^3}{2\mu R}\sin(\omega t)-2\sum_{n=1}^{\infty}\frac{J_1(rr_n)\Omega}{\mu(r_n^2)J_1(Rr_n)}$$

$$\times\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t\sin(\omega s)G_{1-\beta,1-\beta-\beta k,k+1}(0,t-s)ds \qquad \text{...(21)}$$

$$\tau(r,t)=\frac{\Omega r^2}{R}Sin(\omega t)+2\sum_{n=1}^{\infty}\frac{J_2(rr_n)\Omega}{(r_n)J_1(Rr_n)}$$

$$\times\sum_{k=o}^{\infty}(-\vartheta r_n^2)^k\int_0^t\sin(\omega s)G_{1-\beta,1-\beta-\beta k,k+1}(0,t-s)ds \qquad \text{...(22)}$$

4.2 Ordinary Second Grade Fluid

Applying $\beta\to1$

$$\omega(r,t)=\frac{\Omega r^3}{2\mu R}\sin(\omega t)-2\sum_{n=1}^{\infty}\frac{J_1(rr_n)\Omega}{\mu(r_n^2)J_1(Rr_n)}$$

$$\times\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t\sin(\omega s)G_{o,-k,k+1}(0,t-s)ds \qquad \text{....(22)}$$

$$\tau(r,t)=\frac{\Omega r^2}{R}Sin(\omega t)+2\sum_{n=1}^{\infty}\frac{J_2(rr_n)\Omega}{(r_n)J_1(Rr_n)}$$

$$\times\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t\sin(\omega s)G_{o,-k,k+1}(0,t-s)ds \qquad \text{...(23)}$$

4 Consluscion

This work presents exact solutions for the velocity field and associated shear stress corresponding to the flow of fractional second grade fluid performing sinusoidal motion in an infinite cylinder. To find the exact solutions, Laplace transformation and finite Hankel transformation along with application of Bessel functions have been used. Governing equations of the fractional fluid of type second grade along with all initial conditions and boundary conditions have been satisfied by the obtained solutions. Generalized $G_{.,.,.}(.,t)$ function appeared as a powerful tool to write velocity field and shear stress in series form. $\alpha\to0$ makes it possible to get Newtonian velocity field and shear stress. Solutions for ordinary second grade fluid are recovered by applying limit $\beta\to1$.

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ISSN 2229-5518

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