Functionals related to a to the Bitrace on Partial *O -Algebras

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ABSTRACT: We consider functionals defined on some certain subspaces of a partial O*-algebra $\mathfrak{M}$ (i.e, a standard, unital, subalgebra, of a partial *- algebras $\mathcal{L}_w^+(\mathcal{D},\mathcal{H})$). On these subspaces we define the right *-representations(resp., left *--representations) and using such representations we introduce the right (resp., left) regular functionals related to the Bitrace. Simple relations are given for such functionals.

Key words: partial *- algebras $\mathcal{L}_w^+(\mathcal{D},\mathcal{H})$, Bitrace, regular functionals, *-representations.

1. *Introduction:*

In recent years algebras of unbounded operators have been studied by many mathematicians. In the algebraic formulation of quantum field theory or quantum statistical mechanics, the $C^* -$ algebraic setting is however too restrictive since in general the observables of a physical system are unbounded linear operators. The C^* - algebraic approach to quantum theory is a rigid scheme to include in its framework all objects of physical interest and this has led to several possible generalizations namely quasi* - algebras, partial *- algebras and so on. Here we consider one of such generalization called the ***partial O *- algebras*** $\mathfrak{M}$. ,Ekhaguere (2007) introduced an unbounded bitrace on a partial O*-algebra $\mathfrak{M}$. The unbounded bitrace played an important role in the classification of partial O*- algebra $\mathfrak{M}$. Here we consider two unbounded functionals $\psi^{\mathcal{G}_\tau^r}_{(.\ ,\ .)}$: $\mathcal{G}_\tau^r \times \mathcal{G}_\tau^r \to \mathbb{C}^*$ and $\psi^{\mathcal{G}_\tau^\ell}_{(.\ ,\ .)}$: $\mathcal{G}_\tau^\ell \times \mathcal{G}_\tau^\ell \to \mathbb{C}^*$, respectively, where, $\mathcal{G}_\tau^r$, $\mathcal{G}_\tau^\ell$, are dense subspaces respectively. The notion of *right* *- *representations* (resp., *left* *-- *representations*) is introduced. With this notions we define *right* (resp., *left*) *regular functionals* related to such bitrace defined on partial O*- algebra

$\mathfrak{M}$. We state the properties of such functionals.

2. ***Preliminaries on Partial *-Algebra:***

The basic structure is a quadruplet $(\mathcal{A}, \Gamma, *, \cdot)$. This comprises of an involutive complex linear space $\mathcal{A}$ with an involution * , and a relation $\Gamma \subseteq \mathcal{A} \times \mathcal{A}$ on $\mathcal{A}$, with a partial multiplication "$\cdot$" on $\mathcal{A}$, such that

1) $(x, y) \in \Gamma \Leftrightarrow x.y \in \mathcal{A}$
2) $(x, y) \in \Gamma \Leftrightarrow (y^*, x^*) \in \Gamma$, and $(x.y)^* = y^*.x^*$;
3) $(x, y) \in \Gamma$ and $(x, z) \in \Gamma \Rightarrow (x, \alpha y + \beta z) \in \Gamma$ and then
$$x.(\alpha y + \beta z) = \alpha(x.y) + \beta(x.z)$$

A partial *-algebra is in general***, non-associative*** thereby making the study largely dependent on several classes of multipliers introduced as follows.For a partial algebra $(\mathcal{A}, \Gamma, *, \cdot)$ for a subset $\mathfrak{S} \subseteq \mathcal{A}$ and a point $x \in \mathcal{A}$, let $L(x) = \{x \in \mathcal{A}: (y, x) \in \Gamma\}$ and $R(x) = \{y \in \mathcal{A}: (x, y) \in \Gamma\}$

$$L(\mathfrak{S}) = \bigcap\{x \in \mathcal{A}: (y, x) \in \Gamma\} = \bigcap L(x)$$

$$R(\mathfrak{S}) = \bigcap\{y \in \mathcal{A}: (x, y) \in \Gamma\} = \bigcap R(x)$$

$$M(\mathfrak{S}) = L(\mathfrak{S}) \bigcap R(\mathfrak{S}).$$

If $\Gamma = \mathcal{A} \times \mathcal{A}$ then the sets reduces to $\mathcal{A}$ and $\mathcal{A}$ is now called a * algebra.

A concrete partial *-algebra arises as follows. Let $\mathcal{D}$ be a complex pre-Hilbert space, with inner product that is assumed to be linear on the right, and norm $\|\cdot\|$, and completion $\mathcal{H}$. We denote by $L^+(\mathcal{D}, \mathcal{H})$ the set of all linear maps A , each with range in $\mathcal{H}$, such that domain $(A) = \mathcal{D}$ and domain $(A^*) \supset \mathcal{D} =$ domain (A). Equipped with the involution $A \longmapsto A^+ = A^* \upharpoonright \mathcal{D}$ and the usual notion of addition and scalar multiplication, $L^+(\mathcal{D}, \mathcal{H})$ is a complex involutive linear space given by the set

$$L^+(\mathcal{D}, \mathcal{H}) = \{A \in L(\mathcal{D}, \mathcal{H}) : \mathcal{D}(A^*) \supset \mathcal{D}\}$$

Let $\Gamma = \{(A, B) \in L^+(\mathcal{D}, \mathcal{H}) \times L^+(\mathcal{D}, \mathcal{H}): \mathrm{B}\mathcal{D} \subset domain(A^{+*}),\ A^*\mathcal{D} \subset domain(B^*)\}$

Then, the relation Γ induces, and is induced by, a partial multiplication "$\cdot$" on $L^+(\mathcal{D}, \mathcal{H})$ given by $A.B = A^{+*}B$ for

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$(A, B) \in L^{+}(\mathcal{D}, \mathcal{H})$. The quadruplet $(L^{+}(\mathcal{D}, \mathcal{H}), \Gamma, *, \cdot)$ is therefore a partial * algebra. We denote it by $L_{W}^{+}(\mathcal{D}, \mathcal{H})$.The set $L^{+}(\mathcal{D}) = \{A \in L^{+}(\mathcal{D}, \mathcal{H})$: range A $\subset \mathcal{D}$, $A^{*}\mathcal{D} \subset \mathcal{D}\}$ is a *-algebra. A subalgebra of $L^{+}(\mathcal{D})$ is called an ***O^{*}- algebra*** on $\mathcal{D}$. While a subalgebra of $L_{W}^{+}(\mathcal{D}, \mathcal{H})$ is called a ***partial O^{*}- algebra*** on $\mathcal{D}$.

Topologies on $\mathcal{M} \subset L_{W}^{+}(\mathcal{D}, \mathcal{H})$ be a partial O*- algebra on $\mathcal{D}$

1. The strong * operator topology is the locally convex topology on $\mathcal{M}$ induced by the semi norm $p^{*}{}_{\xi}(x)$ defined on $\mathcal{M}$ by $p^{*}{}_{\xi}(x) = \|x\xi\| + \|x^{+}\xi\|$, with $x \in \mathcal{M}$, $\xi \in \mathcal{D}$
2. The weak operator topology is induced by the family of semi norms $\{p_{\xi,\eta}\}$ defined on $\mathcal{M}$ by $p_{\xi,\eta}(x) = \langle x\xi, \eta\rangle$, with $x \in \mathcal{M}, \xi, \eta \in \mathcal{D}$.
3. Let $\mathcal{D}^{\infty} = \{\{\xi_n\} \subset \mathcal{D}: \sum(\|\xi_n\|^2 + \|\xi_n\|^2) < \infty, x \in \mathcal{M}\}$, such that $\{\xi_n\}, \{\eta_n\} \subset \mathcal{D}$. The σ –weak operator topology is the locally convex topology induced by seminorm $\{p_{\xi_n,\eta_n}\}$ defined on $\mathcal{M}$ by $p_{\xi_n,\eta_n}(x) = \sum|\langle x\xi_n, \eta_n\rangle|$, with $x \in \mathcal{M}$.

Let $\mathcal{M}$ be a partial O*- algebra on $\mathcal{D}$ and $\|\xi\|_x = \|x\xi\|$, with $x \in \mathcal{M}$. Let $t_{\mathcal{M}}$ be the locally convex topology on $\mathcal{D}$ generated by the seminorms $\{\|\xi\|_x : x \in \mathcal{M}\}$. We have the following : A partial O*- algebra on $\mathcal{D}$ is called *closed* if the locally convex space $(\mathcal{D}, t_{\mathcal{M}})$ is complete and is called *standard if* $\mathcal{M}$ *is* closed and $\overline{x^{+}} = x^{*}$, for each $x \in \mathcal{M}$.

Ideals: Let $\mathcal{M}$ be a partial O*- algebra on $\mathcal{D}$ and $\mathcal{B}$ a subspace of $\mathcal{M}$. Then $\mathcal{B}$ is a left ideal (resp., a right ideal; resp., an ideal) of $\mathcal{M}$ if $L(\mathcal{M}).\mathcal{B} \subseteq \mathcal{B}$ (resp., $\mathcal{B}.R(\mathcal{M}) \subseteq \mathcal{B}; resp.,$ $\mathcal{B}$ is both a *left* ideal and *right* ideal).

Bitrace : Let $\mathcal{M}$ be a unital partial O *- algebra on domain $\mathcal{D}$, with unit e, and

$$\mathcal{M}_{+} = \{x \in \mathcal{M}: \langle \xi, x\xi\rangle \geq 0, \forall \xi \in \mathcal{D}\},$$

let the set of all maps $\varphi: \mathcal{M} \times \mathcal{M} \to C^{*}$ be denoted by wgt $(\mathcal{M})$ satisfying the following properties;

a) $\varphi(x, \alpha y) = \alpha\varphi(x, y), \alpha \in \mathbb{C}, x, y \in \mathcal{M}$, with $0 \cdot (\pm\infty) = 0$;

b) $\varphi(x, y) = \varphi(\overline{y, x}), \quad x, y \in \mathcal{M}$,

c) $\varphi(x.y, z) = \varphi(y, x^+.z), \quad x, y, z \in \mathcal{M}$, with $x \in L(y), x^+ \in L(z)$

d) $\varphi(x, x) \in \mathbb{R}_+ \cup \{+\infty\}, \quad x \in \mathcal{M}$,

e) $\varphi(e, x) \in \mathbb{R}_+ \cup \{+\infty\}, \quad x \in \mathcal{M}_+$

f) $\varphi(e, x + y) = \varphi(e, x) + \varphi(e, y), \quad x, y \in \mathcal{M}_+$

a member of wgt $(\mathcal{M})$ will be called a weight on $\mathcal{M}$. A pair $(\tau, \mathcal{N}_\tau)$ will be called a ***bitrace*** on $\mathcal{M}$ provided that

i) $\tau \in$ wgt $(\mathcal{M})$

ii) $\tau(x, y) = \tau(y^+, x^+)$, $x, y \in \mathcal{M}$

iii) $\mathcal{N}_\tau$ is an ideal of $\mathcal{M}$

iv) The restriction of τ to $\mathcal{N}_\tau \times \mathcal{N}_\tau$ is a positive sesquilinear form on $\mathcal{N}_\tau$.

***** *-Representations***

A *- representation of a partial *- algebra $\mathcal{A}$ is a * homomorphism of $\mathcal{A}$ into $L_W^+(\mathcal{D}, \mathcal{H})$ satisfying $\pi(e) = 1$ whenever $e \in \mathcal{A}$, that is,

i) π is linear

ii) $x \in L(y)$ in $\mathcal{A}$ implies $\pi(x) \in L(\pi(y))$ and $\pi(x).\pi(y) = \pi(xy)$

iii) $\pi(x^*) = (\pi(x))^+$ for $x \in \mathcal{A}$

A faithful homomorphism if $x \in \mathcal{A}$ and $\pi(0) = 0 \Rightarrow x = 0$. A faithful homomorphism π from $\mathcal{A}_1 \longrightarrow \mathcal{A}_2$ whose inverse π^{-1} is homomorphism from $\mathcal{A}_2 \longrightarrow \mathcal{A}_1$ is called an ***isomorphism***

3 Functionals Determined By A Bitrace On A Partial O- Algebras*

Here we consider two unbounded functionals $\psi^{\mathcal{G}_\tau^r}_{(\cdot\,,\,\cdot)}: \mathcal{G}_\tau^r \times \mathcal{G}_\tau^r \to \mathbb{C}^*$ and $\psi^{\mathcal{G}_\tau^\ell}_{(\cdot\,,\,\cdot)}: \mathcal{G}_\tau^\ell \times \mathcal{G}_\tau^\ell \to \mathbb{C}^*$, respectively, where, $\mathcal{G}_\tau^r$, $\mathcal{G}_\tau^\ell$, are dense subspaces respectively. The notion of *right* *-

representations (resp., *left* *--*representations*) is introduced. With this notions we defined *right* (resp., *left*) *regular functionals.* The *regularity* of the functionals depends on that of the bitrace. We state the properties of such functionals.

Left , Right And Regular Bitrace (Resp., Representation):

Let $\mathfrak{M}$ be a partial O* algebra and $(\tau, \mathcal{N}_\tau)$ a Bitrace on $\mathfrak{M}$. We introduce the following two closed ideals related to $\mathcal{N}_\tau$ (*the definition ideal*) of τ as follows; let x_r, x_ℓ be nonzero elements of $\mathfrak{M}$ respectively, such that, $x_r \neq e$, $x_\ell \neq e$, where e is the unit element of $\mathfrak{M}$, then for any two nonzero elements $a \in \mathcal{L}(\mathfrak{M})$, $b \in \mathcal{R}(\mathfrak{M})$, such that $a \neq e, b \neq e,$ the sets

$\mathcal{N}_\tau^\ell = \{x_\ell \in \mathfrak{M} : \tau(a.x_\ell, a.x_\ell) < \infty,\ a \in \mathcal{L}(\mathfrak{M})\ \}$,

$\mathcal{N}_\tau^r = \{x_r \in \mathfrak{M} : \tau(x_r.b, x_r.b) < \infty,\ b \in \mathcal{R}(\mathfrak{M})\ \}$,

are called the left (resp., right) *ideals of* $\mathfrak{M}$. Where $\mathcal{L}(\mathfrak{M})$ is the set of left multiplier of $\mathfrak{M}$ and $\mathcal{R}(\mathfrak{M})$ is the set of right multipliers of $\mathfrak{M}$. We define quotient maps on these ideals. Hence for the left ideal (resp., right ideal) we have the corresponding subspaces defined as

$\mathcal{J}_\tau^\ell = \left\{x \in \mathcal{N}_\tau^\ell : \tau(a.x, a.x) = 0, a \in \mathcal{L}(\mathfrak{M})\ \right\}$

$\mathcal{J}_\tau^r = \{x \in \mathcal{N}_\tau^r : \tau(x.b, x.b) = 0,\ b \in \mathcal{R}(\mathfrak{M})\ \}$.

The quotient maps $\lambda_\tau^\ell : \mathcal{N}_\tau^\ell \to \mathcal{N}_\tau^\ell / \mathcal{J}_\tau^\ell$, $\lambda_\tau^r : \mathcal{N}_\tau^r \to \mathcal{N}_\tau^r / \mathcal{J}_\tau^r$ are given by $\lambda_\tau^\ell(x_\ell) = x_\ell + \mathcal{J}_\tau^\ell$ and $\lambda_\tau^r(x_r) = x_r + \mathcal{J}_\tau^r$. Let $\left[\lambda_\tau^\ell(\mathcal{N}_\tau^\ell)\right]$, $[\lambda_\tau^r(\mathcal{N}_\tau^r)]$ be the linear spans of $\lambda_\tau^\ell\left(\mathcal{N}_\tau^\ell\right)$, $\lambda_\tau^r(\mathcal{N}_\tau^r)$ respectively, and let the action of a sesquilinear form on both the subspaces, be given by,

$$\langle \lambda_\tau^\ell(x_\ell), \lambda_\tau^\ell(y_\ell) \rangle = \tau(x_\ell, y_\ell),\quad x_\ell,\ y_\ell \in \mathcal{N}_\tau^\ell \tag{1}$$

$$\langle \lambda_\tau^r(x_r), \lambda_\tau^r(y_r) \rangle = \tau(x_r, y_r),\quad x_r, y_r \in \mathcal{N}_\tau^r. \tag{2}$$

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Naturally, this inner product induces a Hilbert space completion for the closed spaces $[\lambda_\tau^\ell(\mathcal{N}_\tau^\ell)]$, $[\lambda_\tau^r(\mathcal{N}_\tau^r)]$. We denote by $\mathcal{H}_\tau^\ell, \mathcal{H}_\tau^r$ their respective Hilbert spaces. We have the following definitions of the left and right regular Bitrace on $\mathfrak{M}$ based on the construct given above.

Definition: 1

Let $\mathcal{N}_\tau^\ell \neq \{0\}$ and let $\mathcal{G}_\tau^\ell$ be a subspace, such that

(i) $\mathcal{G}_\tau^\ell \subset \mathcal{L}(\mathfrak{M}) \cap \mathcal{N}_\tau^\ell$

(ii) The linear span $[\lambda_\tau^\ell(\mathcal{G}_\tau^\ell)]$ of $\lambda_\tau^\ell(\mathcal{G}_\tau^\ell)$ is dense in $\mathcal{H}_\tau^\ell$, and is denoted by $\mathcal{D}_\tau^\ell$

(iii) $\mathcal{G}_\tau^\ell$ is a core for $\tau_{/\mathcal{D}_\tau^\ell}$.

(iv) A bitrace defined on $\mathfrak{M}$ satisfying $\tau(a_1.x_1, a_2.x_2) = \tau\big(a_1,\ a_2.(x_1^+.x_2)\big) < \infty$, is called a *left regular bitrace*, where $a_1, a_2 \in \mathcal{G}_\tau^\ell$, $x_1, x_2 \in \mathfrak{M}$ and with $a_2 \in \mathcal{L}(x_1^+.x_2)$, $x_1^+ \in \mathcal{L}(x_2)$,.

Definition: 1′

For $\mathcal{N}_\tau^r \neq \{0\}$, let $\mathcal{G}_\tau^r$ be a subspace, such that

(i) $\mathcal{G}_\tau^r \subset \mathcal{R}(\mathfrak{M}) \cap \mathcal{N}_\tau^r$

(ii) the linear span $[\lambda_\tau^r(\mathcal{G}_\tau^r)] \equiv \mathcal{D}_\tau^r$ of $\lambda_\tau^r(\mathcal{G}_\tau^r)$ is dense in $\mathcal{H}_\tau^r$ and is denoted by $\mathcal{D}_\tau^r$

(iii) $\mathcal{G}_\tau^r$ is a core for $\tau_{/\mathcal{D}_\tau^r}$.

(iv) A bitrace on $\mathfrak{M}$ satisfying $\tau(w_1.b_1, w_2.b_2) = \tau(b_1.,(w_2.w_1^+).b_2) < \infty$ is called a *right regular bitrace* where, $b_1, b_2 \in \mathcal{G}_\tau^r$, $w_1, w_2 \in \mathfrak{M}$ with $b_2 \in \mathcal{R}(w_2.w_1^+)$, $w_1^+ \in \mathcal{R}(w_2)$

Definition:2

A left (*resp.*, right) , regular representation on a partial O*- algebra $\mathfrak{M}$, denoted by , π_τ^ℓ (*resp.*, π_τ^r), is defined

(i) for any $x_1 \in \mathfrak{M}$ and $a_1 \in \mathcal{G}_\tau^\ell$

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$\pi_\tau^\ell(x_1)\lambda_\tau^\ell(a_1) = \lambda_\tau^\ell(a_1.x_1).$

(3)

(ii) for any $w_1 \in \mathfrak{M}$ and $b_1 \in \mathcal{G}_\tau^r$

$$\pi_\tau^r(w_1)\lambda_\tau^r(b_1) = \lambda_\tau^r(w_1.b_1)$$

$(3)'$

Remark:1

If a representation π is both left and right regular with domain then, it is called a regular representation.

Functionals Determined By Bitraces:

The two functionals, $\psi_{(.\,,\,.)}^{\mathcal{G}_\tau^r}, \psi_{(.\,,\,.)}^{\mathcal{G}_\tau^\ell}$ introduced, called the right functional (resp., left functional) defined on $\mathcal{G}_\tau^r \times \mathcal{G}_\tau^r$ (resp., $\mathcal{G}_\tau^\ell \times \mathcal{G}_\tau^\ell$) are implemented by representations. Let $x \in \mathfrak{M}$ such that $x \neq e$, and let $x_1 \in \mathcal{L}(\mathcal{G}_\tau^\ell)$ and $x_2 \in \mathcal{R}(\mathcal{G}_\tau^r)$, then for arbitrary $a \in \mathcal{G}_\tau^\ell$, $b \in \mathcal{G}_\tau^r$, let $x \to a.x$, $x \to x.b$ be continuous maps with respect to the locally convex topology t_m (***the graph topology)*** such that $a.x \equiv a_1 \in \mathcal{G}_\tau^\ell$ and $x.b \equiv b_1 \in \mathcal{G}_\tau^r$, we have $x_1.a_1 \in \mathcal{G}_\tau^\ell$ and $b_1.x_2 \in \mathcal{G}_\tau^r$, since $\tau(x_1.a_1, x_1.a_1) < \infty$ and $\tau(b_1.x_2, b_1.x_2) < \infty$. These representations on the dense subspaces $\mathcal{G}_\tau^r, \mathcal{G}_\tau^\ell$, denoted by $\pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)}$(resp., $\pi_\tau^{\mathcal{L}(\mathcal{G}_\tau^\ell)}$), is defined by

$$\pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)}(x_2)\lambda_\tau^r(x.b) = \pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)}(x_2)\lambda_\tau^r(b_1) = \lambda_\tau^r\big((x.b).x_2\big) = \lambda_\tau^r(b_1.x_2) \qquad (4)$$

$$\pi_\tau^{\mathcal{L}(\mathcal{G}_\tau^\ell)}(x_1)\,\lambda_\tau^\ell(a.x) = \pi_\tau^{\mathcal{L}(\mathcal{G}_\tau^\ell)}(x_1)\,\lambda_\tau^\ell(a_1) = \lambda_\tau^\ell\big(x_1.(a.x)\big) = \lambda_\tau^\ell(x_1.a_1) \qquad (4)'$$

for $x_2 \in \mathcal{R}(\mathcal{G}_\tau^r)$ (resp., $x_1 \in \mathcal{L}(\mathcal{G}_\tau^\ell)$)

Remark:2

We note that λ_τ^r, $(resp.,\ \lambda_\tau^\ell)$ acts on the elements of $\mathcal{G}_\tau^\ell$ (resp., $\mathcal{G}_\tau^r$) by a flip action

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given by $\lambda_\tau^\ell(x_1.a_1) = \lambda_\tau^r(a_1^+.x_1^+)$, $\lambda_\tau^r(b_1.x_2) = \lambda_\tau^\ell(x_2^+.b_1^+)$, respectively.

Definition:3

Using these representations in, (4), $(4)'$ we define the right (resp., left) functionals as mappings $\psi_{(.\,,\,.)}^{\mathcal{G}_\tau^r}: \mathcal{G}_\tau^r \times \mathcal{G}_\tau^r \to \mathbb{C}^*$ and $\psi_{(.\,,\,.)}^{\mathcal{G}_\tau^\ell}: \mathcal{G}_\tau^\ell \times \mathcal{G}_\tau^\ell \to \mathbb{C}^*$ by,

$$\psi_{(x_1,a_1)}^{\mathcal{G}_\tau^r}(b_1,x_2) = \langle \pi_\tau^{\mathcal{L}(\mathcal{G}_\tau^\ell)}(x_1)\lambda_\tau^\ell(a.x),\ \pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)}(x_2)\lambda_\tau^r(x.b)\rangle,\quad x \in \mathfrak{M} \qquad (5)$$

$$\psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}(x_1,a_1) = \langle \pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)}(x_2)\lambda_\tau^r(x.b), \pi_\tau^{\mathcal{L}(\mathcal{G}_\tau^\ell)}(x_1)\,\lambda_\tau^\ell(a.x)\rangle,\quad x \in \mathfrak{M} \qquad (5)'$$

for each $(x_1,a_1) \in \mathcal{G}_\tau^\ell \times \mathcal{G}_\tau^\ell$ and $(b_1,x_2) \in \mathcal{G}_\tau^r \times \mathcal{G}_\tau^r$. The name right (resp., left) functionals arises from the representation appearing on the right. The sesquilinear forms are assumed to be linear on the right . In simple terms the functionals are given by,

$$\psi_{(x_1,a_1)}^{\mathcal{G}_\tau^r}(b_1,x_2) = \langle \lambda_\tau^r(a_1^+.x_1^+), \lambda_\tau^r(b_1.x_2)\rangle = \tau(a_1^+.x_1^+, b_1.x_2) < \infty$$

$$\psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}(x_1,a_1) = \langle \lambda_\tau^\ell(x_2^+.b_1^+), \lambda_\tau^\ell(x_1.a_1)\rangle = \tau(x_2^+.b_1^+, x_1.a_1) < \infty \qquad (6)$$

Remark: 3 For the unit element, we have

$$\psi_{(x_1,a_1)}^{\mathcal{G}_\tau^r}(e.e) \equiv \psi_{(x_1,a_1)}^{\mathcal{G}_\tau^r} ;$$

$$\psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}(e.e) \equiv \psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}$$

Definition:4

A left functional $\psi^{\mathcal{G}_\tau^\ell}$, is a *left regular* functional if for any $a_1, a_2 \in \mathcal{G}_\tau^\ell$, and $y_1, y_2 \in \mathcal{R}(\mathcal{G}_\tau^\ell)$, with $y_1^+ \in \mathcal{L}(y_2)$, $(y_1^+.y_2) \in \mathcal{R}(a_2)$ the functional is of the form $\psi_{a_1,e}^{\mathcal{G}_\tau^\ell}(e, a_2.(y_1^+.y_2)) < \infty$, whenever the defining bitrace is also left regular. Similarly we have the *right regular* functional to be of the form $\psi_{b_1,e}^{\mathcal{G}_\tau^r}(e, (x_2.x_1^+).b_2) < \infty$, where $b_1, b_2 \in \mathcal{G}_\tau^r$, and $x_1, x_2 \in \mathcal{L}(\mathcal{G}_\tau^r)$,

$x_1^+ \in \mathcal{R}(x_2)$ with $(x_2.x_1^+) \in \mathcal{L}(b_2)$.

Definition:5

We called ψ a regular functional if for any, $a_1, a_2 \in \mathcal{G}_\tau \subset \mathcal{M}(\mathfrak{M}) \cap \mathcal{N}_\tau$, and

$y_1, y_2 \in \mathcal{M}(\mathcal{G}_\tau)$, with $(y_1.y_2^+) \in \mathcal{L}(a_2^+)$, $(y_1^+.y_2) \in \mathcal{R}(a_2)$

we have $\psi^{\mathcal{G}_\tau^\ell}_{a_1,e}(e, a_2.(y_1^+.y_2)) = \psi^{\mathcal{G}_\tau^r}_{a_1^+,e}(e,(y_1.y_2^+).a_2^+) < \infty$,

The following lemma give the relations between the left and right functionals.

Lemma 1

(a) $\psi^{\mathcal{G}_\tau^\ell}_{y_1^+.a_1^+}(a_2, y_2)$ is a left regular functional, whenever τ is left regular

(b)

$$\psi^{\mathcal{G}_\tau^r}_{x_1,a_1}(b_1, x_2) = \psi^{\mathcal{G}_\tau^\ell}_{b_1^+,x_2^+}(a_1^+, x_1^+)$$

Proof;

(a) Let $a_1, a_2 \in \mathcal{G}_\tau^\ell$, $y_1, y_2 \in \mathcal{R}(\mathcal{G}_\tau^\ell)$, and $({y_1}^+.y_2) \in \mathcal{R}(a_2)$. since τ is assumed to be left regular, we need only to show that

$$\psi^{\mathcal{G}_\tau^\ell}_{{y_1}^+,a_1^+}(a_2, y_2) = \psi^{\mathcal{G}_\tau^\ell}_{a_1}(a_2.({y_1}^+.y_2)),$$

$$\psi^{\mathcal{G}_\tau^\ell}_{{y_1}^+,a_1^+}(a_2, y_2) = \langle \lambda_\tau^r({y_1}^+.a_1^+), \lambda_\tau^\ell(a_2.y_2)\rangle$$

$$= \langle \lambda_\tau^\ell(a_1.y_1), \lambda_\tau^\ell(a_2.y_2)\rangle$$

$$= \tau(a_1.y_1, a_2.y_2)$$

$$= \tau\big(a_1, a_2.({y_1}^+.y_2)\big)$$

$$= \langle \lambda_\tau^\ell(a_1.e), \lambda_\tau^\ell\big(a_2.({y_1}^+.y_2)\big)\rangle$$

$$= \psi^{\mathcal{G}_\tau^\ell}_{a_1.e}(a_2.({y_1}^+.y_2)) = \psi^{\mathcal{G}_\tau^\ell}_{a_1}(a_2.({y_1}^+.y_2))$$

(b)

$$\psi^{\mathcal{G}_\tau^r}_{(x_1,a_1)}(b_1, x_2) = \langle \lambda_\tau^\ell(x_1.a_1), \lambda_\tau^r(b_1.x_2)\rangle$$

$$= \langle \lambda_\tau^r(a_1^+.x_1^+), \lambda_\tau^r(b_1.x_2)\rangle$$

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$= \tau(a_1^+ . x_1^+, b_1 . x_2)$

$= \tau(x_2^+ . b_1^+, x_1 . a_{1.})$

$= \langle \lambda_\tau^r(x_2^+ . b_1^+), \lambda_\tau^r(x_1 . a_1), \rangle$

$= \langle \lambda_\tau^r(x_2^+ . b_1^+), \lambda_\tau^\ell(a_1^+ . x_1^+), \rangle = \psi_{(b_1^+, x_2^+)}^{\mathcal{G}_\tau^\ell}(a_1^+, x_1^+)$

Remark:4

From this lemma, we have the following relations,

$\psi_{(y_1{}^+, a_1^+)}^{\mathcal{G}_\tau^\ell}(a_2, y_2) = \psi_{(y_2^+, a_2^+)}^{\mathcal{G}_\tau^r}(a_1, y_1 \ ),$ (7)

$\psi_{(b_1, x_2)}^{\mathcal{G}_\tau^\ell}(x_1 . a_1) = \psi_{(x_1^+, b_1^+)}^{\mathcal{G}_\tau^r}(a_1^+ . x_2^+).$

These expressions are analogous to commutations relations of the right and left representations on a generalized Hilbert algebras: This provides us with the following;

Proposition:1

If τ is a regular bitrace on $\mathcal{G}_\tau$, then ψ is a regular functional on $\mathcal{G}_\tau$

Proof;

Suppose that τ is regular on $\mathcal{G}_\tau \subset \mathcal{M}(\mathfrak{M}) \cap \mathcal{N}_\tau$, to show that the functional is regular we need only to show that $\psi_{a_1}^{\mathcal{G}_\tau^\ell}(a_2 . (y_1^+ . y_2)) = \psi_{a_1^+}^{\mathcal{G}_\tau^r}((y_1 . y_2^+) . a_2^+) < \infty$.

Let $(y_2 . y_1^+) \in \mathcal{L}(a_1)$, and $(y_2 . y_1^+)^+ \in \mathcal{L}(a_2^+)$

$\psi_{a_1}^{\mathcal{G}_\tau^\ell}(a_2 . (y_1^+ . y_2)) = \psi_{a_{1.}.e}^{\mathcal{G}_\tau^\ell}(a_2 . (y_1^+ . y_2) = \langle \lambda_\tau^r(a_{1.} . e), \lambda_\tau^\ell((a_2 . (y_1^+ . y_2)) \rangle$

$= \langle \lambda_\tau^\ell(a_1^+ . e), \lambda_\tau^\ell((a_2 . (y_1^+ . y_2)) \rangle$

$= \tau(a_1^+, \ a_2 . (y_1^+ . y_2) \ )$

$= \tau(a_1^+ . y_1, \ a_2 . y_2 \ )$

$= \tau(y_2^+ . a_2^+, y_1^+ . a_1 \ )$

$= \tau(a_2^+, \ (y_2 . y_1^+) . a_1 \ )$

$=$ $\tau((y_2.y_1^+)^+.a_2^+,\ a_1)$

$=$ $\tau(a_1^+,\ a_2.(y_2.y_1^+))$

$=$ $\langle\lambda_\tau^\ell(a_1^+),\lambda_\tau^r\big((y_1.y_2^+).a_2^+\big)\rangle$

$=$ $\psi_{a_1^+}^{\mathcal{G}_\tau^r}((y_1.y_2^+).a_2^+) < \infty$

Lemma: 2 For $a_1 \in \mathcal{G}_\tau^\ell,\ b_1 \in \mathcal{G}_\tau^r$, we have, $\pi_\tau^{\mathcal{L}(\mathcal{G}_\tau^\ell)}(\mathcal{G}_\tau^\ell)_\sigma \subset \pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)}(\mathcal{G}_\tau^r)'_\sigma$

Proof;

The proof is based on noting that commutations is implied by the relation,

$$\psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}(x_1,a_1) = \psi_{(x_1^+,b_1^+)}^{\mathcal{G}_\tau^r}(a_1^+,x_2^+).$$

Let $x_2 \in \mathcal{R}(\mathcal{G}_\tau^r)\cap\mathcal{L}(\mathcal{G}_\tau^\ell)$, $x_1 \in \mathcal{L}(b_1^+)$ and $(x_2.a_1) \in \mathcal{G}_\tau^\ell$, we have,

$\psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}(x_1,a_1) =$

$\langle\pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)}(x_2)\lambda_\tau^r(b_1),\pi_\tau^{\mathcal{L}(\mathcal{G}_\tau^\ell)}(x_1)\ \lambda_\tau^\ell(a_1)\rangle$

$= \langle\lambda_\tau^r(b_1.x_2),\lambda_\tau^\ell(x_1.a_1)\rangle$

$= \langle\lambda_\tau^\ell(x_2^+.b_1^+),\lambda_\tau^\ell(x_1.a_1)\rangle$

$= \tau(x_2^+.b_1^+,x_1.a_1)$

$= \tau(x_1^+.b_1^+,\ x_2.a_1)$

$= \langle\lambda_\tau^\ell(x_1^+.b_1^+),\lambda_\tau^\ell(x_2.a_1)\rangle$

$=\langle\lambda_\tau^\ell(x_1^+.b_1^+),\lambda_\tau^r(a_1^+.x_2^+)\rangle$

$= \langle\pi_\tau^{\mathcal{L}(\mathcal{G}_\tau^\ell)}(x_1^+)\lambda_\tau^\ell(b_1^+),\pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)}(x_2^+)\lambda_\tau^r(a_1^+)\rangle$

$= \psi_{(x_1^+,b_1^+)}^{\mathcal{G}_\tau^r}(a_1^+,x_2^+).$

Remark:5

The functional $\psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}$ is an idempotent by composition that is,

$$\psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}\circ\psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell} = \psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}(x_1,a_1).$$

Proof:

$\psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}\circ\psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}(x_1,a_1)$

$= \psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}\left(e,\psi_{(b_1,x_2)}^{\mathcal{G}_\tau^\ell}(x_1,a_1)\right)$

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$= \psi^{\mathcal{G}_\tau^r}_{(x_1^+,b_1^+)}\left(e, \psi^{\mathcal{G}_\tau^r}_{(x_1^+,b_1^+)}(a_1^+, x_2^+)\right)$

$=\langle \pi_\tau^{\mathcal{L}(\mathcal{G}_\tau^\ell)}(x_1^+)\lambda_\tau^\ell(b_1^+),\ \pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)} \circ$

$\psi^{\mathcal{G}_\tau^r}_{(x_1^+,b_1^+)}(a_1^+, x_2^+)\lambda_\tau^r(e.e)\rangle$

$= \langle \lambda_\tau^\ell(x_1^+.b_1^+),\ \langle \lambda_\tau^\ell(x_1^+,b_1^+),\ \pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)} \circ$

$\lambda_\tau^r(a_1^+.x_2^+)\,\rangle\, \lambda_\tau^r(e.e)\rangle$ (**)

note that, for $\lambda_\tau^r(e.e) = I$, from definition we have,

$\pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)} \circ \lambda_\tau^r(a_1^+.x_2^+)$

$= \pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)}\big(\lambda_\tau^r(a_1^+.x_2^+)\big)\lambda_\tau^r(e.e) =$

$\lambda_\tau^r(e.e)\lambda_\tau^r(a_1^+.x_2^+) = \lambda_\tau^r(a_1^+.x_2^+)$,

hence eqn. (**) becomes,

$\psi^{\mathcal{G}_\tau^\ell}_{(b_1,x_2)} \circ \psi^{\mathcal{G}_\tau^\ell}_{(b_1,x_2)}(x_1,a_1) = \langle \lambda_\tau^\ell(x_1^+.b_1^+),$

$\langle \lambda_\tau^\ell(x_1^+,b_1^+), \lambda_\tau^r(a_1^+.x_2^+)\,\rangle \lambda_\tau^r(e.e)\rangle,$

$=$

$\langle \lambda_\tau^\ell(x_1^+,b_1^+), \lambda_\tau^r(a_1^+.x_2^+)\,\rangle\ \langle \lambda_\tau^\ell(x_1^+.b_1^+), \lambda_\tau^r(e.e)\rangle$

$= \psi^{\mathcal{G}_\tau^r}_{(x_1^+,b_1^+)}(a_1^+, x_2^+)\ \psi^{\mathcal{G}_\tau^r}_{(x_1^+,b_1^+)}(e,e)$

$= \psi^{\mathcal{G}_\tau^\ell}_{(b_1,x_2)}(x_1,a_1)$

Summary of some properties of Functionals determined By Bitraces:

The functionals $\psi^{\mathcal{G}_\tau^r}_{(.\,,\,.)}$ and $\psi^{\mathcal{G}_\tau^\ell}_{(.\,,\,.)}$ called the right (resp., left) functional defined on $\mathcal{G}_\tau^r \times \mathcal{G}_\tau^r$ (resp., $\mathcal{G}_\tau^\ell \times \mathcal{G}_\tau^\ell$) satisfies the following properties;

(i) $\psi^{\mathcal{G}_\tau^\ell}_{y_1{}^+.a_1^+}(a_2, y_2)$ is a left regular functional, whenever τ is left regular

(ii) $\psi^{\mathcal{G}_\tau^r}_{x_1,a_1}(b_1,x_2) = \psi^{\mathcal{G}_\tau^\ell}_{b_1^+,x_2^+}(a_1^+,x_1^+)$ $\psi^{\mathcal{G}_\tau^\ell}_{(y_1{}^+,a_1^+)}(a_2,y_2) =$ $\psi^{\mathcal{G}_\tau^r}_{(y_2^+,a_2^+)}(a_1,y_1\ ),$

(Commutations relations)

(iii) If τ is a regular bitrace on $\mathcal{G}_\tau$, then ψ is a regular functional on $\mathcal{G}_\tau$.

(iv) For $a_1 \in \mathcal{G}_\tau^\ell$, $b_1 \in \mathcal{G}_\tau^r$, we have, $\pi_\tau^{\mathcal{L}(\mathcal{G}_\tau^\ell)}(\mathcal{G}_\tau^\ell)_\sigma \subset \pi_\tau^{\mathcal{R}(\mathcal{G}_\tau^r)}(\mathcal{G}_\tau^r)'_\sigma$.

(v) $\psi^{\mathcal{G}_\tau^\ell}_{(b_1,x_2)} \circ \psi^{\mathcal{G}_\tau^\ell}_{(b_1,x_2)}(x_1,a_1) =$ $\psi^{\mathcal{G}_\tau^\ell}_{(b_1,x_2)}(x_1,a_1)$.

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