$for\, r = r_1, r_2, \dots, r_n \to 1-$ combining $(a)$ and $(b)$ we obtain $k = \rho_g^p\big(f(r_1, r_2, \dots, r_n)\big)$

hence proof the theorem.

### *SUM AND PRODUCT THEOREM*

***Theorem* 2.** In the unit disc U, having $f_1$ and $f_2$ of generalized relative orders $\rho_g^p\big(f_1(r_1, r_2, \dots, r_n)\big)$ and $\rho_g^p\big(f_2(r_1, r_2, \dots, r_n)\big)$ respectively, where $g$ is entire having the property (R) then

$(i)$ $\rho_g^p\big(f_1(r_1, r_2, \dots, r_n) + f_2(r_1, r_2, \dots, r_n)\big) \le max\{\rho_g^p\big(f_1(r_1, r_2, \dots, r_n)\big), \rho_g^p\big(f_2(r_1, r_2, \dots, r_n)\big)\}$

$(ii)$ $\rho_g^p\big(f_1(r_1, r_2, \dots, r_n). f_2(r_1, r_2, \dots, r_n)\big) \le max\{\rho_g^p\big(f_1(r_1, r_2, \dots, r_n)\big), \rho_g^p\big(f_2(r_1, r_2, \dots, r_n)\big)\}$

the some inequality holds for quotients the equality holds in $(ii)\, if\, \rho_g^p\big(f_1(r_1, r_2, \dots, r_n)\big) \ne \rho_g^p\big(f_2(r_1, r_2, \dots, r_n)\big)$.

***Proof*.** Let $\rho_1 = \rho_g^{[p]}\big(f_1(r_1, r_2, \dots, r_n)\big)$ and $\rho_2 = \rho_g^{[p]}\big(f_2(r_1, r_2, \dots, r_n)\big)$ and $\rho_1 \le \rho_2$. We assume that $\rho_g^{[p]}\big(f_1(r_1, r_2, \dots, r_n)\big)$ and $\rho_g^{[p]}\big(f_2(r_1, r_2, \dots, r_n)\big)$ both are finite because if one of them or both are infinite inequality are evident for arbitrary $\varepsilon > 0$ and for all $r_1, r_2, \dots, r_n, 0 < r_1, r_2, \dots, r_n < 1$, sufficiently close to 1 we have

$$T_{f_1}(r_1, r_2, \dots, r_n)$$

$$< T_g\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_1+\varepsilon}\right)$$

$$\le \log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_1+\varepsilon}\right)$$

and

$$T_{f_2}(r_1, r_2, \dots, r_n)$$

$$< T_g\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)$$

$$\le \log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)$$

Using lemma 2 for all $r_1, r_2, \dots, r_n, 0 < r_1, r_2, \dots, r_n < 1$, sufficiently close to 1

$$T_{f_1 \ne f_2}(r_1, r_2, \dots, r_n)$$

$$\le T_{f_1}(r_1, r_2, \dots, r_n) \pm T_{f_2}(r_1, r_2, \dots, r_n) + O(1)$$

$$\le \log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_1+\varepsilon}\right) + \log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right) + O(1)$$

$$\le 3 \log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)$$

$$= \frac{1}{3} \log\left[G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)\right]^9$$

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$$\leq \frac{1}{3}\log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)^{\sigma}$$

by lemma 1, for any $\sigma > 1$

$$\leq T_g\left(2\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)^{\sigma}\right)$$

by lemma 2, since

$$T_g^{-1}T_{f_1\neq f_2}(r_1,r_2,\dots,r_n) \leq$$

$$\log 2 + \log\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)^{\sigma}$$

$$\leq \sigma exp^{[p-2]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon} + O(1)$$

$$\log^{[2]} T_g^{-1}T_{f_1\neq f_2}(r_1,r_2,\dots,r_n) \leq exp^{[p-3]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon} + O(1)$$

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)+f_2(r_1,r_2,\dots,r_n)\right)$$

$= \lim_{r_1,r_2,\dots,r_n\to 1-} sup\, \frac{\log^{[P]} T_g^{-1}T_{f_1\neq f_2}(r_1,r_2,\dots,r_n)}{-\log(1-r_1)(1-r_2)\dots(1-r_n)} \leq \rho_2+\varepsilon$

since $\varepsilon > 0$ is arbitrary,

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)+f_2(r_1,r_2,\dots,r_n)\right) \leq \rho_2$$

$$\leq max\left\{\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right),\rho_g^{[p]}\left(f_2(r_1,r_2,\dots,r_n)\right)\right\}$$

which proves $(i)$, $for$ $(ii)$, since

$T_{f_1\cdot f_2}(r_1,r_2,\dots,r_n) \leq T_{f_1}(r_1,r_2,\dots,r_n) + T_{f_2}(r_1,r_2,\dots,r_n)$

we obtain similarly as above

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n).f_2(r_1,r_2,\dots,r_n)\right)$$

$$\leq max\left\{\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right),\rho_g^{[p]}\left(f_2(r_1,r_2,\dots,r_n)\right)\right\}$$

Let $f = f_1f_2$ and

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right) < \rho_g^{[p]}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

Then applying $(ii)$, we have

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right) \leq \rho_g^{[p]}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

again since $f_2 = \frac{f}{f_1}$, applying the first part of $(ii)$, we have

$$\rho_g^{p}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

$$\leq max\left\{\rho_g^{[p]}\left(f(r_1,r_2,\dots,r_n)\right),\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right)\right\}$$

since

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right) < \rho_g^{p}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

we have

$$\rho_g^{[p]}\left(f(r_1,r_2,\dots,r_n)\right) \leq \rho_g^{p}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

$$= max\{\rho_g^{p}\left(f_1(r_1,r_2,\dots,r_n)\right),\rho_g^{p}\left(f_2(r_1,r_2,\dots,r_n)\right)\}$$

when

$$\rho_g^{p}\left(f_1(r_1,r_2,\dots,r_n)\right) \neq \rho_g^{p}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

this prove the theorem.

RELATIVE ORDER WITH RESPECT TO THE DERIVATIVE OF AN ENTIRE FUNCTIONS

***Theorem* 3**. In the unit disc, $f$ is analytic function and $g$ be transcendental entire having the property $(R)$, then

$$\rho_g^{[p]}\left(f(r_1,r_2,\dots,r_n)\right) = \rho_{g'}^{[p]}\left(f(r_1,r_2,\dots,r_n)\right)$$

where $g'$ denotes the derivative of $g$ . To prove the theorem we require the following lemmas.

***Lemma* 3**. [1] If $g$ be transcendental entire, then for all $r_1,r_2,\dots,r_n, 0<r_1,r_2,\dots,r_n<1$, sufficiently close to 1 for any $\lambda>0$

$$T_{g'}\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda},\dots,\frac{1}{(1-r_n)^\lambda}\right)$$

$$\le 2T_g\left(2\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda},\dots,\frac{1}{(1-r_n)^\lambda}\right)\right)$$

$$+O\left(T_g\left(2\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda},\dots,\frac{1}{(1-r_n)^\lambda}\right)\right)\right)$$

***Lemma* 4**. [1] If $g$ be transcendental entire, then for all $r_1,r_2,\dots,r_n, 0<r_1,r_2,\dots,r_n<1$, sufficiently close to 1 for any $\lambda>0$

$$T_g\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda},\dots,\frac{1}{(1-r_n)^\lambda}\right)$$

$$\le \alpha_0\left[T_{g'}\left(2\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda},\dots,\frac{1}{(1-r_n)^\lambda}\right)\right)\right]$$

$$+\log\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda},\dots,\frac{1}{(1-r_n)^\lambda}\right)$$

Where $\alpha_0$ is constant which is only dependent on $g(0)$.

$PROOF\ OF\ THE\ THEOREM$

***Proof***. We obtain for $r_1,r_2,\dots,r_n, 0< r_1,r_2,\dots,r_n<1$, sufficiently close to 1 from the lemma 3 and lemma 4.

$(c)$

$$T_{g'}\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda},\dots,\frac{1}{(1-r_n)^\lambda}\right)$$

$$<[c]T_g\left(2\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda},\dots,\frac{1}{(1-r_n)^\lambda}\right)\right)$$

and

$(d)$

$$T_g\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda},\dots,\frac{1}{(1-r_n)^\lambda}\right)$$

$$<[c_0]T_{g'}\left(2\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda},\dots,\frac{1}{(1-r_n)^\lambda}\right)\right)$$

Where $c_0$ and $\lambda>0$ be any number from the definition of $\rho_{g'}^{[p]}\left(f(r_1,r_2,\dots,r_n)\right)$ , we get for any arbitrary $\varepsilon>0$

$$T_f(r_1,r_2,\dots,r_n) < T_{g'}\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda},\dots,\frac{1}{(1-r_n)^\lambda}\right)^{\rho_{g'}f(r_1,r_2,\dots,r_n)+\varepsilon}\right)$$

for all $r_1,r_2,\dots,r_n, 0<r_1,r_2,\dots,r_n<1$, from $(c)$ and by lemma 1 and lemma 2

for all $r_1,r_2,\dots,r_n, 0<r_1,r_2,\dots,r_n<1$, sufficiently close to 1

$$T_f(r_1,r_2,\dots,r_n) < [c]T_g\left(2exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_{g'}^{[p]}f(r_1,r_2,\dots,r_n)+\varepsilon}\right)$$

$$\le [c]\log G\left(2exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_{g'}^{[p]}f(r_1,r_2,\dots,r_n)+\varepsilon}\right)$$

$$= \frac{1}{3}\log\left[G\left(2exp^{[p-1]}\begin{pmatrix}\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\\ \dots,\frac{1}{(1-r_n)}\end{pmatrix}^{\rho_{g'}^{[p]}f(r_1,r_2,\dots,r_n)+\varepsilon}\right)\right]^{3[c]}$$

$$\leq \frac{1}{3}\log\left(G\left(2exp^{[p-1]}\begin{pmatrix}\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\\ \dots,\frac{1}{(1-r_n)}\end{pmatrix}^{\rho_{g'}^{[p]}f(r_1,r_2,\dots,r_n)+\varepsilon}\right)\right)^{\sigma}$$

For any $\sigma > 1$

$$\leq T_g\left(2^{\sigma+1}\left(exp^{[p-1]}\begin{pmatrix}\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\\ \dots,\frac{1}{(1-r_n)}\end{pmatrix}^{\rho_{g'}^{[p]}f(r_1,r_2,\dots,r_n)+\varepsilon}\right)^{\sigma}\right)$$

$$\rho_g^{[p]}\big(f(r_1,r_2,\dots,r_n)\big)$$

$$= \lim_{r_1,r_2,\dots,r_n\to 1-} sup\frac{\log^{[P]} T_g^{-1}\,T_f(r_1,r_2,\dots,r_n)}{-\log(1-r_1)(1-r_2)\dots.(1-r_n)}$$

$$\leq \rho_{g'}^{[p]}\big(f(r_1,r_2,\dots,r_n)\big)+\varepsilon$$

since $\varepsilon > 0$ is arbitrary, so

$$\rho_g^{[p]}\big(f(r_1,r_2,\dots,r_n)\big) \leq \rho_{g'}^{[p]}\big(f(r_1,r_2,\dots,r_n)\big)$$

from (d) we obtain similarly,

$$\rho_{g'}^{[p]}\big(f(r_1,r_2,\dots,r_n)\big) \leq \rho_g^{[p]}\big(f(r_1,r_2,\dots,r_n)\big)$$

so

$$\rho_g^{[p]}\big(f(r_1,r_2,\dots,r_n)\big) = \rho_{g'}^{[p]}\big(f(r_1,r_2,\dots,r_n)\big)$$

Hence prove the theorem.

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