ISSN 2229-5518

Logical sets of quantum operators

Nikolay Raychev

Abstract - A central component of the quantum calculations is the logical summary of the classic concept for the operators for identity and negation. The characteristic of the quantum states and operators, based on this summary, is in the basis of the logic of the formalized qubit operators. In this report is examined the general idea for the states of identity and negation. Also specific examples from these classes are discussed.

Index Terms— Boolean function, circuit, composition, encoding, gate, phase, quantum.

1 Introduction

The process of the quantum calculations can be modeled through the use of a Hilbert space, where the state of a system of qubits is described through a vector unit of length and the operators of these states retain the length, or unitary operators of the space. The mathematics of the Hilbert spaces is an important prerequisite for operation with Quantum calculations. The characteristic of the unitary operators is vital, in order to be proceed in ways at which the unitary operators to be set as a combination of other operators, as it provides various ways for verification, whether a given operator really is unitary. A key element of the logical sets for quantum operators deals with states, which have probably equivalent measurement results in comparison with their physical equivalent states.

2 Identity and negation at quantum calculations

2.1. Identity

In the classic case the identity of a state of a given bit can be accepted as leaving this state unchanged. One of the ways for examining the identity at the quantum calculations is by accepting that the state of a *n*-qubit system is identical when its behavior remains unchanged at measurement. Therefore, the operators for classical identity could be a summary through a set of *n*-qubit operators, which do not change the behavior of one *n*-qubit state at measurement in the computational basis.

Definition 1 *Operator A is a n-qubit* ***operator for extensional identity****, if for all n-qubit states* $|\phi\rangle$

$$\boldsymbol{P}[|\phi\rangle \rightarrow |b\rangle] = \boldsymbol{P}[\boldsymbol{A}|\phi\rangle \rightarrow |b\rangle] \quad (1)$$

for each $b \in \mathbb{B}^n$. *State* $A|\phi\rangle$ *is* ***the extensional equivalent*** *of* $|\phi\rangle$ *and* Ext_n *identifies the set of all the operators for extensional identity.*

From Definition 1 follows that the only difference between a state and its extensional equivalent are the phases of the probability amplitudes, and not their values. Intuitively can be thought for operators for extensional identity as operators, which change the phase of the probability amplitudes for zero or more basic states. Both the *Z* gate of Pauli and *-I* have to be single operators for extensional identity. The *Z* operator of Pauli introduces a phase change to the probability amplitude of $|1\rangle$, while the operator *-I* introduces a global phase change. The equivalency of states, which differ only by a global phase factor, has never been in question, so that the identity of the subset from operators for extensional identity, which perform a similar phase change, leads to a further improvement of the summarized operators for identity.

Definition 2 *Operator A is a n-qubit* ***operator for intensional identity****, if for all n-qubit states* $|\phi\rangle$

$$\langle\phi|\boldsymbol{A}|\phi\rangle = \pm\mathbf{1} \quad (2)$$

State $A|\phi\rangle$ *is* ***the intensional equivalent*** *of* $|\phi\rangle$ *and* Int_n *identifies the set of all the operators for intensional identity.*

The exact characteristic of Int_n follows from Definition 2.

Formal prerequisite 1 $Int_n = \{I_n, -I_n\}$

Proof. If $A \in Int_n$ such that $\langle\phi|A|\phi\rangle = 1$. Then $A|\phi\rangle = |\phi\rangle$ and therefore A must be the *n*-cubit operator for identity I_n. Similarly, when $\langle\phi|A|\phi\rangle = -1$, then $A|\phi\rangle = -|\phi\rangle$ and therefore A is $-I_n$. In this way $Int_n \subseteq \{I_n, -I_n\}$. On the other hand, $\langle\phi|I_n|\phi\rangle = \langle\phi|\phi\rangle = 1$ and therefore $I_n \in Int_n$. Similarly, $\langle\phi|-I_n|\phi\rangle = (-1)\langle\phi|\phi\rangle = -1$ and therefore $-I_n \in Int_n$.

The characteristic of Ext_n is given by a Formal prerequisite 2 below.

Formal prerequisite 2

$$\mathrm{Ext}_n = \left\{A = \sum_{b=0}^{2^n-1} A_{b,b}|b\rangle\langle b| : \forall b \in \mathbb{B}^n, A_{b,b} \in \{-1,1\}\right\} \quad (3)$$

Proof. If at first is determined that all the operators from the form

$$\sum_{b=0}^{2^n-1} A_{b,b}\,|b\rangle\langle b|$$

are operators for extensional identity. It is accepted that

$$U \in \left\{A = \sum_{b=0}^{2^n-1} A_{b,b}|b\rangle\langle b| : \forall b \in \mathbb{B}^n, A_{b,b} \in \{-1,1\}\right\}$$

In order for U to be an operator for extensional identity, must be shown that for each $b \in \mathbb{B}^n$ $\langle b|\phi\rangle^2 = \langle b|U|\phi\rangle^2$.

$$U|\phi\rangle = \sum_{b=0}^{2^n-1} U_{b,b}\langle b|\phi\rangle\,|b\rangle$$

$$= \sum_{b=0}^{2^n-1} U_{b,b}\phi_b\langle b|b\rangle\,|b\rangle$$

$= \sum_{b=0}^{2^n-1} U_{b,b}\phi_b|b\rangle$ (4)

If it is given that $U_{b,b} \in \{-1,1\}$, then it follows that $|\langle b|U|\phi\rangle| = |\langle b|\phi\rangle|$ for each $b \in \mathbb{B}$. Then it must be true that $\langle b|U|\phi\rangle^2 = \langle b|\phi\rangle^2$ and by Definition 1 $U \in Ext_n$.
On the other hand, all the operators in Ext_n must be with the following form

$$\sum_{b=0}^{2^n-1} A_{b,b}\,|b\rangle\langle b|$$

If it is accepted that $A \in Ext_n$, then it follows that for a basis $|x\rangle$

$$A|x\rangle = \sum_{i=0}^{2^n-1} A_{i,x}\,|i\rangle$$

$$= A_{x,x}|x\rangle \sum_{i\neq x} A_{i,x}|i\rangle$$

$$= \pm|x\rangle$$

If $A_{x,x} \neq 1$, then because A is unitary, there must be some non-empty subset of the residual from column x with non-null values and so $A|x\rangle \neq \pm x$. From here follows that $A_{x,x} \in \{-1,1\}$ for each $x \in \mathbb{B}^n$. So all the operators Ext_n are 2^n x 2^n diagonal matrices with entries ± 1.
It is clear that Int_n is a subset of Ext_n.

Formal consequence 1 $Int_n \subset Ext_n$.
Proof. The operators I_n и $-I_n$ obviously give rise to states, which at measurement behave in the same way as their input values, and therefore $Int_n \subset Ext_n$. From Formal prerequisite 2 follows that there are operators in Ext_n, which change the phase only to a subset from the probability amplitudes. For example for n = 1 Z the operator of Pauli is in Ext_n and is not equivalent neither to *I*, nor to – *I*. Therefore must be true that $Int_n \subset Ext_n$. In this way can be said that Ext_n is the set from the *n*-qubit operators, which identify *n*-qubit states in such way that their behavior at measurement stays unchanged. The set Int_n is a subset of Ext_n, which raises states physical equivalent, to a global phase factor, and $Ext_n \setminus Int_n$ is the set of operators, which introduce the phase changes in comparison with a non-empty subset of the main states.

Note 1 *In the above paragraph, and in the context of the rest of this operation the symbol \ is used for designating the difference between the two sets. In other words, $A\setminus B$ contains all the elements of A, which are not part from B.*
The set Ext_n can be designated through the function *Id* : $\mathbb{B}^{2^n} \to Ext_n$, where for $b \in \mathbb{B}^{2^n}$

$$Id_b = \sum_{i=0}^{2^n-1} -1^{b_i}\,|i\rangle\langle i| \quad (5)$$

This allows for easy enumeration of *n*-qubit operators for identity. In addition, it is convenient that the negation of operator Id_b is equivalent to Id_b

$$-Id_b = Id_{\bar{b}} = \sum_{i=0}^{2^n-1} -1^{\bar{b}_i}\,|i\rangle\langle i| \quad (6)$$

2.2 Negation

Unlike the identity, which leaves the bits unchanged, the negation reverse them. Here again it is possible the negation to be consider in respect of measurement and actual quantum states.

Definition 3 *The Operator A is a n-qubit operator for **extensional negation**, if for all n-qubit states* $|\phi\rangle$

$$\boldsymbol{P}[|\phi\rangle \to |b\rangle] = \boldsymbol{P}[\boldsymbol{A}|\phi\rangle \to |\bar{b}\rangle] \quad (7)$$

for each $b \in \mathbb{B}^n$. *The state* $A\,|\phi\rangle$ *is **the extensional negation** of* $|\phi\rangle$ *and* $Next_n$ *designates the set of all operators for extensional negation.*

The extensional negation requires the amplitudes of a given basic state and its negation to be swapped, but still allows for the occurrence of random phase changes. Similar to the characteristic of Ext_n is also the characteristic of $Next_n$.

Formal prerequisite 3.

$$Next_n = \left\{A = \sum_{b=0}^{2^n-1} A_{b,\bar{b}}|b\rangle\langle\bar{b}| : \forall b \in \mathbb{B}^n, A_{b,\bar{b}} \in \{-1,1\}\right\} \quad (4.7)$$

Where $\bar{b}$ is the negation of the *n*-bit binary representation of *b*.
Proof. If

$$A = A \in \sum_{b=0}^{2^n-1} A_{b,\bar{b}}|b\rangle\langle\bar{b}|,\ A_{b,\bar{b}} \in \{-1,1\}$$

Then $|\psi\rangle = A|\phi\rangle$, $\psi_i = \langle A_i|\phi\rangle = \phi_{\bar{\imath}}$. From this follows that $\boldsymbol{P}[|\phi\rangle \to |b\rangle] = \boldsymbol{P}[|\psi\rangle \to |\bar{b}\rangle]$

and therefore $A \in Next_n$.
If $A \in Next_n$, then for a basis $|x\rangle$

$$A|x\rangle = \sum_{i=0}^{2^n-1} A_{i,x}\,|i\rangle$$

$$= A_{\bar{x},x}|\bar{x}\rangle \sum_{i\neq x} A_{i,x}|i\rangle$$

$$= \pm|\bar{x}\rangle$$

If $A_{\bar{x}x} \neq \pm 1$, then $A|x\rangle$ obviously can not to be $|\bar{x}\rangle$, because other non-empty subset of column x must be non-null, because A *is* unitary. From here follows, that $A_{\bar{x}x} \in \{-1,1\}$ or the equivalent $A_{x,\bar{x}} \in \{-1,1\}$. The physical analogue of the negation could in its turn be considered as orthogonality. First is defined a set from *n*-qubit operators, which connect states with orthogonal states.

Definition 4 *The Operator A is a n-qubit Ortho operator, if for all n-qubit states* $|\phi\rangle$,

$$\langle\phi|\boldsymbol{A}|\phi\rangle = \boldsymbol{0} \quad (8)$$

The set of such operators is designated with $Ortho_n$.

Formal prerequisite 4 characterizes the operators in $Ortho_n$.

Formal prerequisite 4 $Ortho_n$ *is the set from operators* $A = A_{i,j=0\ldots 2^n-1}$, *such that:*

1. $A_{i,j} \in \{-1,0,1\}$
2. $\sum_j |A_{i,j}| = 1$
3. $A_{i,j} + A_{j,i} = 0 \; \forall i,j$

Proof. First it is proven that each matrix *A*, *satisfying* the above conditions, is unitary.

In fact

$$(AA^\dagger)_{i,j} = \sum_{k=0}^{2^n-1} A_{i,k} A^\dagger{}_{k,j}$$

$$= \sum_{k=0}^{2^n-1} A_{i,k} A_{j,k}$$

For i = j

$$(AA^\dagger)_{i,j} = \sum_{k=0}^{2^n-1} |A_{i,k}| = 1$$

For i ≠ j

$$(AA^\dagger)_{i,j} = \sum_{k=0}^{2^n-1} A_{i,k} A_{j,k}$$

and when $k = i$ (similarly $k = j$), then $A_{i,i} A_{j,k} = 0$. For any other $i \neq j$, $A_{i,k} A_{j,k} = 0$. If $A_{i,k} \neq 0$ and $A_{j,k} \neq 0$, then from condition three follows, that $A_{k,i} \neq 0$ and $A_{k.j} \neq 0$ and therefore $\sum_{j=0}^{2^n-1} |A_{k,j}| > 1$, which is contrary to condition two. If it is demonstrated that, for each matrix *A*, satisfying the above conditions $\langle\phi|A|\phi\rangle = 0$ for all *n*-qubit states $|\phi\rangle$ and so $A \in Ortho_n$. If the state $|\phi\rangle = \sum_{i=0}^{2^n-1} \phi_i |i\rangle$, then the inner product $\langle\phi|A|\phi\rangle = \sum_{ij} A_{i,j} \phi_j \phi_i$. From here

$$\langle\phi|A|\phi\rangle = \sum_{i,j} A_{i,j} \phi_j \phi_i = \sum_{i,j} A_{i,i} {\phi_i}^2 + \sum_{(i,j) i \neq j} A_{i,j} \phi_j \phi_i$$

$$= \sum_{i,j} A_{i,i} {\phi_i}^2 + \sum_i \sum_{i>j} (A_{i,j} + A_{j,i}) \phi_j \phi_i = 0 \qquad (9)$$

where the last equality withstands, because $A_{i,j} = 0$ and $A_{i,j} + A_{j,i} = 0$ and therefore $A \in Ortho_n$. On the other hand it is accepted that $A \in Ortho_n$. The purpose is to be proven that all three conditions above are true. Then, it follows that $\langle x|A|x\rangle = 0$ for each basic state $|x\rangle$. It is known that $A_{x,x} = 0$ and that there is some *j*, such that $A_{j,x} = \pm 1$, otherwise $|\phi\rangle = A|x\rangle$ must be a superposition of the basic states and $\langle x|A|x\rangle \neq 0$. Thus it is seen that conditions one and two must be true for $A \in Ortho_n$. Moreover, it is known that for $A \in Ortho_n$, $A_{i,i} = 0$ for each $i \in \mathbb{B}^n$. That is why equation 4.8 is limited to

$$\sum_i \sum_{i>j} (A_{i,j} + A_{j,i}) \phi_j \phi_i = 0 \qquad (10)$$

When condition three withstands, then equation 4.9 is obviously true for each |ϕ⟩. In case that some $k > 1$ instances of $A_{i,j}$ and $A_{j,i}$ the pairs does not satisfy condition three, then equation 10 is limited to $\sum_{i=1}^{k} 2A_{ij,ji} \phi_{ij} \phi_{ii} = 0$ and is true only for a certain |ϕ⟩, which is contrary to the definition for operators in the set $Ortho_n$. Therefore condition three must be true for each $A \in Ortho_n$. Formal consequence 2 shows that the orthogonality in no case does not ensure extensional negation: The extensional negation requires the probability for measurement of a certain basis to be reversed with the probability for the logical negation of this basis, while the orthogonality generally allows arbitrary pairs of the basis to exchange the probabilities. This is visible from the structure of $Ortho_n$ the operators.

Formal consequence 2 *For n > 1,*

1. $Next_n \setminus Orto_n \neq \emptyset$
2. $Orto_n \setminus Next_n \neq \emptyset$

Proof. From Formal prerequisite 4 is known, that for n > 2) can be built $A \in Ortho_n$ such that $A_{i,\bar{\imath}} = A_{\bar{\imath},i} = 0$. Then follows that $Ortho_n \setminus Next_n \neq \emptyset$, as A is in $Ortho_n$, and not in $Next_n$. For example let's $A \in Ortho_n$ be defined as follows:

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It is clear that *A* corresponds to the conditions, given in Formal prerequisite 4, and that $\langle\phi|U|\phi\rangle = \phi_0\phi_2 - \phi_1\phi_3 - \phi_2\phi_0 + \phi_3\phi_1 = 0$. But $U|\phi\rangle$ is not an extensional negation of $|\phi\rangle$, as $P[|\phi\rangle \to |0\rangle] = P[U|\phi\rangle \to |3\rangle]$ only when $\phi_1 = \phi_0$, which is not for each $|\phi\rangle$, as required by the definition for $Next_n$. From Formal prerequisite 3 is known, that for $n \geq 1$ can be built $A \in Next_n$ such that $A_{\bar{\imath},i} + A_{i,\bar{\imath}} \neq 0$. Such operator is in $Next_n$, and not $Ortho_n$. It then follows that for $n > 1$ $Ortho_n \setminus Next_n \neq \emptyset$. For $n = 1$ the operator *X is* an example of such an operator.

To be compatible with the summary of the identity, the operators for intensional negation will be limited to the subset of $Next_n$, which intersects with the $Ortho_n$.

Definition 5 *The Operator A is a n-qubit operator for* ***intensional negation*** *, if* $A \in Nint_n = Ortho_n \cap Next_n$. *The state* $A|\phi\rangle$ *e* ***the intensional negation*** *of* $|\phi\rangle$.

The characterization of $Nint_n$ the operators follows from their definition and the characterization of $Ortho_n$ and $Next_n$.

Formal consequence 3

$$Nint_n = \left\{ A = \sum_{i=0}^{2^n-1} (-1)^{b_i} |i\rangle\langle\bar{\imath}| : b_i \oplus b_{\bar{\imath}} = 0 \right\}$$

Proof. The proof follows from Definition 5 and formal prerequisites 3 and 4. Since it is clear that $Nint_n$ is strict subset both of $Ortho_n$, and of $Next_n$, this can be stated in Formal prerequisite 5 for completeness.

Formal prerequisite 5 *For n > 1,*

1. $Nint_n \subset Ortho_n$
2. $Nint_n \subset Next_n$

Proof. From definition 4 is seen that $Nint_n \subseteq Next_n$ and $Nint_n \subseteq Ortho_n$. The Operator $N \otimes N$ is in $Next_n$, and not $Nint_n$ and therefore $Nint \subset Next$. And finally, from consequence 2 is known, that there are $Ortho_n$ operators, which fall outside $Next_n$ and therefore may not be in $Nint_n$. The set is

ISSN 2229-5518

$Next_n$ presented through $Neg : \mathbb{B}^{2^n} \rightarrow Next_n$, where for $b \in \mathbb{B}^{2^n}$

$$Neg_b = Id_b\left(\sum_{i=0}^{2^n-1} |i\rangle\langle\bar{\iota}|\right) = \sum_{i=0}^{2^n-1} -1^{b_i} |i\rangle\langle\bar{\iota}| \quad (11)$$

While the negation of operator Neg_b follows from the negation of Id_b.

$$Neg_{\bar{b}} = Id_{\bar{b}} \sum_{i=0}^{2^n-1} |i\rangle\langle\bar{\iota}| = -Neg_b \quad (12)$$

Examples

If first is examined the space of the single qubit operators, since they serve as elementary constructive elements for the operators over many qubits. Equation 13 lists the full set from single qubit operators.

$$\mathrm{Id}_{00} = |0\rangle\langle 0| + |1\rangle\langle 1| = I$$

$$\mathrm{Id}_{01} = |0\rangle\langle 0| - |1\rangle\langle 1| = Z$$

$$\mathrm{Id}_{10} = -|0\rangle\langle 0| + |1\rangle\langle 1| = -Z$$

$$\mathrm{Id}_{11} = -|0\rangle\langle 0| - |1\rangle\langle 1| = -I$$

$$\mathrm{Neg}_{00} = |0\rangle\langle 1| + |1\rangle\langle 0| = X$$

$$\mathrm{Neg}_{01} = |0\rangle\langle 1| - |1\rangle\langle 0| = -N$$

$$\mathrm{Neg}_{10} = -|0\rangle\langle 1| + |1\rangle\langle 0| = N$$

$$\mathrm{Neg}_{11} = -|0\rangle\langle 1| - |1\rangle\langle 0| = -X \quad (13)$$

Apparently Id_{00} and Id_{11} are the single qubit operator for identity I and its negation and therefore form Int_1. The operator is Id_{01} the Z operator of Pauli and Id_{10} is its negation. These four operator form the entire Ext_1. Addressing to $Next_1$, is visible, that the operator Neg_{00} is the X operator of Pauli and Neg_{11} is its negation. The operators Neg_{01} and Neg_{10} are rotational operators from $Nint_1$ and can be described in terms of the operators of Pauli as – $N = ZX$ and $N = XZ$ respectively. With regard to the sets from operators for identity and negation for $n = 1$, is seen the following allocation of operators.

$$Int_1 = \{\pm I\}$$

$$Nint_1 = Ortho_1 = \{\pm N\}$$

$$\mathrm{Ext}_1 = Int_1 \cup \{\pm Z\}$$

$$Next_1 = Nint_1 \cup \{\pm X\} \quad (14)$$

While $Ortho_1 = Nint_1$, consequence 3 shows that this does not extend beyond $n = 1$. In example 1 are given three examples for $Next_2$ and $Ortho_2$ operators.

Example 1

$$N \otimes I = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in Ortho_2$$

$$N \otimes N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in Next_2 \backslash Ortho_2$$

$$N \otimes X = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in Nint_2$$

3 Conclusion

The summarization of the operators for identity and negation, presented in this report, provides the logical basis for formalization of qubit operators. The identity is generally caught through the operators in the set Ext_n, as defined in Definition 2 and characterized in Formal prerequisite 2. These operators leave the extent of the probability amplitudes, associated with each basic state, unchanged, while possibly change the phase. On the other hand, the operators in $Next_n$, which characterize the negation, as defined in Definition 3 and characterized in Formal prerequisite 3, swap the amplitudes of the probability amplitudes of basis $|i\rangle$ and $|\bar{\iota}\rangle$ for each $i \in \mathbb{B}^n$. Stronger forms of identity rely on the exact quantum state and are characterized by the inner product.