Numerical bifurcation of a host-parasite model undergoing a variation in the growth rate of the host population

Agwu, I. A, Ekaka-a, E.N , Galadima, I.J, Inyama, S.C , Esekhaigbe, E

Abstract: A numerical analysis of a dynamic ecological system is an important tool. In ecological modeling, the behavior of the model is usually studied by a means of bifurcation analysis. This standard technique involves the analysis of qualitative changes in the stability of the system. In this paper, we study the bifurcation of the trivial steady state solution of a host – parasite model due to a variation in the growth rate of the host population when other model parameters are fixed. The critical values of the steady state solution due to fundamental changes in model parameters are obtained. It is found that the stability behavior of the trivial solution changes at this bifurcation point. We also found that as the trivial steady – state solution persists through the bifurcation point p = 0, d = 0, a_2 = 0, this steady – state solution changes from a stable node to a cusp behavior and to a saddle node

Keywords: Host – parasite, bifurcation analysis, stability, steady state.

1. INTRODUCTION

When species interact the population dynamics of each species is affected. [1] proposed a mathematical model which describes host-parasite interactions. In ecological modeling, the behaviour of the model is usually studied by a means of bifurcation analysis. [2] Studied the behaviour of an ecological model using the concept of a bifurcation analysis. [3] Reviewed the use of bifurcation theory to analyse non- linear dynamical systems. They found that bifurcation analysis gives regimes in the parameter space with quantitatively different asymptotic dynamic behaviour of the system. [4] Studied the repercussion of the ecological setting on the outcome of conditional or variable interactions by means of a model that incorporates density-dependent interaction coefficients. Bifurcation analysis of their model show that these dynamics are modelled by ecological factors that are: intrinsic to the association (concerning the sensitivity of the interaction) and extrinsic to the association (the quality of the environment referred to each species alone). [5] formulated epidemiological models for the transmission of a pathogen that can mutate in the host to create a second infectious mutant strain. They found that under certain circumstances, there is a Hopf bifurcation where the endeamic equilibrium loses its stability and periodic solutions appear. [6] Considered the dynamics of a two stage population model. A numerical study of their work reveals a rich bifurcation structure, originating from a degenerate Bogdanov-Takens (BT) bifurcation point. [7] Quantify the dynamics of prion infection. They conducted a bifurcation analysis of the model with respect to two biochemical parameters: the rate of spontaneous transformation and the rate of output of the infectious isoform. They found that the bi-stability properties evidenced by Laurent are confined to a certain range of parameters and those permanent oscillations of the two isoforms concentrations are possible. Their study also predicts that the output rate of the infectious isoform is relatively insensitive to variation of model parameters. [8] Considered ecological consequences of global bifurcation in food chain models using the analyses of four previously published ecological ODE food chain models. They showed how global bifurcations in these models are related to each other and local bifurcations, and used the Shil'nikov homoclinic bifurcation functions as organizing centre of chaos in three dimensional food chain models as an example in their consideration of ecological implications of the global bifurcation. Other works on bifurcation analysis include [9] and [10].In this paper, we study the bifurcation of the trivial steady state solution of Host- Parasite model undergoing a variation in the growth rate of the host population when other model parameters are fixed using the model of [1]. By trivial steady state we mean that which describes the situation with no host and no parasite.

2. HOST-PARASITE MODEL

A lot of Host-Parasite models have been developed, but in this study, we shall consider the model of [1].The model consists of three nonlinear differential equations which describe the dynamics of a Host, Parasite and Top host-population. In this model, they made the following assumptions:

(1) The number of host consumed per unit time by the parasite is proportional to the existing population of the host and the parasite
(2) The population of host grows exponentially in the absence of the parasite.
(3) The population of the parasite decays exponentially in the absence of the host population.
(4) The parasite population switches to an alternative food option as when it faces difficulty to find its favourite host.

ISSN 2229-5518

TABLE 2.1:
NOTATIONS: VARIABLE DESCRIPTION [1]

H(t): Host population at time t
P(t): Parasites population at time t
T(t): Top host population at time t

TABLE 2.1.2:
PARAMETERS DESCRIPTIONS [1]

a_1 = growth rate of the Host Population
d = decay rate of the Parasite Population
a_2 = growth rate of the Top Host Population

$b_i, i=1,2$ measur the consumption effect of parasite population on the Host populations H(t) and top host population T(t) respectively.

$c_i, i=1,2$ measures the proportion of consumed host, H(t) and top host, T(t) respectively that becomes new parasite biomass, P.

b_3 and b_4 Measure the rate of interactions of the host population, H(t) and the top host population respectively

$\beta = \frac{d}{a_1}$ is the ratio of the decay rate of parasite population to the growth rate of the Host population

$\alpha = \frac{d}{a_2}$ is the ratio of the decay rate of parasite population to the growth rate of the top host population.

$\gamma = \frac{a_2}{a_1}$ is the ratio of the growth rate of top host population to the host population.

$\theta = \frac{a_1 b_2}{a_2 b_1}$ is the product of the ratio of growth rate of the host population to the growth rate of the top host and the measure of the consumption effect of parasite population on hosts population.

$\phi = \frac{b_4 d}{a_2 c_1}$ is the product of the ratio of the rate of interaction of top host and decay rate of parasite population to growth rate of top host population and the measure of consumed (parasitized) host population.

Base on the above assumptions, [1] formulated the model given below.

$$
\begin{aligned}
\frac{dH}{dt} &= (a_1 - b_1 P - b_3 T) H \\
\frac{dP}{dt} &= (-d + c_1 H + c_2 T) P \qquad (2.0) \\
\frac{dT}{dt} &= (a_2 - b_2 P - b_4 H) T
\end{aligned}
$$

where a_1, a_2, b_1, b_2, b_3, b_4 c_1, c_2, d, β, α, ϕ, θ, $\gamma > 0$ are the model parameters. The description of variables and each parameter in (2.0) is given above:

2.2 Steady State Analysis

2.2.1 Linearization

In this section we will assume that F_1, F_2 and F_3 are continuously and partially differentiable functions of H,P and T at an arbitrary steady state solution, since this well established result is not the focus of our analysis.

Linearizing the model using $a_1 = \alpha p$, p>0 and α a positive constant of proportionality, gives

$$J = \begin{pmatrix} \alpha p - b_1 P_e - b_3 T_e & -b_1 H_e & -b_3 H_e \\ c_1 P_e & -d + c_1 H_e + c_2 T_e & c_2 P_e \\ -b_4 T_e & -b_2 T_e & a_2 - b_2 P_e - b_4 H_e \end{pmatrix}$$

At (0, 0, 0)

$$J_0 = \begin{pmatrix} \alpha p & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & a_{2e} \end{pmatrix}$$

The eigenvalues of J_0 are $\lambda_1 = \alpha p$, $\lambda_2 = -d$ and $\lambda_3 = a_2$

Without bifurcation analysis, the trivial steady state is unstable since λ_1 and λ_3 are none negative.

3. BIFURCATION ANALYSIS

In this section, we consider 27 instances of bifurcation behaviour at the trivial steady state solution: changing parttern of eigenvalues

1. $\lambda_1 < 0, \quad \lambda_2 < 0, \quad \lambda_3 < 0$
2. $\lambda_1 < 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0$
3. $\lambda_1 < 0, \quad \lambda_2 = 0, \quad \lambda_3 < 0$
4. $\lambda_1 = 0, \quad \lambda_2 < 0, \quad \lambda_3 < 0$
5. $\lambda_1 < 0, \quad \lambda_2 > 0, \quad \lambda_3 = 0$
6. $\lambda_1 < 0, \quad \lambda_2 = 0, \quad \lambda_3 > 0$
7. $\lambda_1 = 0, \quad \lambda_2 < 0, \quad \lambda_3 > 0$
8. $\lambda_1 > 0, \quad \lambda_2 < 0, \quad \lambda_3 = 0$
9. $\lambda_1 > 0, \quad \lambda_2 = 0, \quad \lambda_3 < 0$
10. $\lambda_1 = 0, \quad \lambda_2 > 0, \quad \lambda_3 < 0$
11. $\lambda_1 < 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0$
12. $\lambda_1 = 0, \quad \lambda_2 < 0, \quad \lambda_3 = 0$
13. $\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 < 0$
14. $\lambda_1 > 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0$
15. $\lambda_1 = 0, \quad \lambda_2 > 0, \quad \lambda_3 = 0$
16. $\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 > 0$

17. $\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_3 < 0$
18. $\lambda_1 > 0, \quad \lambda_2 < 0, \quad \lambda_3 > 0$
19. $\lambda_1 < 0, \quad \lambda_2 > 0, \quad \lambda_3 > 0$
20. $\lambda_1 < 0, \quad \lambda_2 < 0, \quad \lambda_3 > 0$
21. $\lambda_1 < 0, \quad \lambda_2 > 0, \quad \lambda_3 < 0$
22. $\lambda_1 > 0, \quad \lambda_2 < 0, \quad \lambda_3 < 0$
23. $\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0$
24. $\lambda_1 > 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0$
25. $\lambda_1 = 0, \quad \lambda_2 > 0, \quad \lambda_3 = 0$
26. $\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 > 0$
27. $\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_3 > 0$

Case 1: Ecological Interpretation

$\lambda_1 < 0 \Rightarrow p < 0$ and $\alpha > 0$ or $p > 0$ and $\alpha < 0$ H-population tend to shrink

$\lambda_2 < 0 \Rightarrow d > 0$ P- population tend to grow

$\lambda_3 < 0 \Rightarrow a_2 < 0$ T- population will tend to shrink

Case 2: Ecological Interpretation

$\lambda_1 < 0 \Rightarrow p <$ H-population tend to shrink

$\lambda_2 = 0 \Rightarrow d = 0$ decay rate of P- population is equal to the growth rate

$\lambda_3 = 0 \Rightarrow a_2 = 0$ growth rate of T- population is equal to the death rate

Case 3: Ecological Interpretation

$\lambda_1 < 0 \Rightarrow p < 0$ H-population tend to shrink

$\lambda_2 = 0 \Rightarrow d = 0$ decay rate of P- population is equal to the growth rate

$\lambda_3 < 0 \Rightarrow a_2 < 0$ T- population will tend to shrink

Case 4: Ecological Interpretation

$\lambda_1 = 0 \Rightarrow p = 0$ growth rate of H- population is equal to the death rate

$\lambda_2 < 0 \Rightarrow d > 0$ P- population tend to grow

$\lambda_3 < 0 \Rightarrow a_2 < 0$ T- population will tend to shrink

TABLE 3.1:
SHOWS OTHER CASES OF BIFURCATION BEHAVIOR AT THE TRIVIAL STEADY STATE.

Cases	λ_1	λ_2	λ_3	Stability behavior
1	$P < 0$	$d > 0$	$a_2 < 0$	Stable

2	$P < 0$	$d = 0$	$a_2 = 0$	Sitting on the cusp
3	$P < 0$	$d = 0$	$a_2 < 0$	Unstable
4	$P = 0$	$d > 0$	$a_2 < 0$	Unstable
5	$P < 0$	$d < 0$	$a_2 = 0$	Unstable
6	$P < 0$	$d = 0$	$a_2 > 0$	Unstable
7	$P = 0$	$d > 0$	$a_2 > 0$	Unstable
8	$P > 0$	$d > 0$	$a_2 = 0$	Unstable
9	$P > 0$	$d = 0$	$a_2 < 0$	Unstable
10	$P = 0$	$d < 0$	$a_2 < 0$	Unstable
11	$P < 0$	$d = 0$	$a_2 = 0$	Sitting on the cusp
12	$P = 0$	$d > 0$	$a_2 = 0$	Sitting on the cusp
13	$P = 0$	$d = 0$	$a_2 < 0$	Sitting on the cusp
14	$P > 0$	$d = 0$	$a_2 = 0$	Sitting on the cusp
15	$P = 0$	$d < 0$	$a_2 = 0$	Sitting on the cusp
16	$P = 0$	$d = 0$	$a_2 > 0$	Sitting on the cusp
17	$P > 0$	$d < 0$	$a_2 < 0$	Unstable
18	$P > 0$	$d > 0$	$a_2 > 0$	Unstable
19	$P < 0$	$d < 0$	$a_2 > 0$	Unstable
20	$P < 0$	$d > 0$	$a_2 > 0$	Unstable
21	$P < 0$	$d < 0$	$a_2 < 0$	Unstable
22	$P > 0$	$d > 0$	$a_2 < 0$	Unstable
23	$P = 0$	$d = 0$	$a_2 = 0$	bifurcation point
24	$P > 0$	$d = 0$	$a_2 = 0$	Sitting on the cusp
25	$P = 0$	$d < 0$	$a_2 = 0$	Sitting on the cusp
26	$P = 0$	$d = 0$	$a_2 > 0$	Sitting on the cusp
27	$P > 0$	$d < 0$	$a_2 > 0$	Unstable

4. DISCUSSION OF CORE RESULTS

Our analysis shows that the trivial steady – state solution is a stable node for $p < 0$, $d > 0$, $a_2 < 0$, sitting on the cusp for ($P < 0$, $d = 0$, $a_2 = 0$), ($P < 0$, $d = 0$, $a_2 = 0$), ($P = 0$, $d > 0$, $a_2 = 0$), ($P = 0$, $d = 0$, $a_2 < 0$), ($P > 0$, $d = 0$, $a_2 = 0$), ($P = 0$, $d < 0$, $a_2 = 0$), ($P = 0$, $d < 0$, $a_2 > 0$), ($P > 0$, $d = 0$, $a_2 = 0$), ($P = 0$, $d < 0$, $a_2 = 0$) and ($P = 0$, $d = 0$, $a_2 > 0$)
and saddle for $p > 0$, $d > 0$, $a_2 > 0$. Therefore as the trivial steady – state solution persists through the bifurcation point $p = 0$, $d = 0$, $a_2 = 0$, this steady – state solution changes from a stable node to a saddle node.

5. CONCLUSION

The concepts of bifurcation is not new, but our numerical method of studying bifurcation of the trivial steady state solution which have not been seen elsewhere provides valuable insights for effective ecosystem functioning and stability
In this study, we have successfully utilized the tool of bifurcation analysis to study the fundamental changes in the qualitative behaviour of stability with respect to the trivial steady state solution due to a variation of a_1, d, and a_2 and their combinations.
The bifurcation analysis of (a_2/b_4, 0, a_1/b_3), (0, $a_2/b_{2,}$ $^d/c_2$) and ($^d/c_1$, $a_1/_{b1,}$ 0) due to a variation in a_1, d, and a_2 and their combinations will be a focus of our further work

REFERENCES

[1] Agwu, I. A, Abdulrahman, S and Nyor, N. (2011). 'Adaptive control of Equilibrium State of Host-parasite model using nonlinear feedback controllers'. Paper presented at the 30th Annual conference of Nigeria Mathematical society(NMS):Recent Advance in Mathematical Science.19 – 22nd july. Minna, Nigeria.(Unpublished)

[2] Troost, T. A; Kooi, S. A and Kooijman, L. M. (2007).'Bifurcation Analysis of ecological and evolutionary processes in Ecosystem'. Journal of ecological Modelling,vol.204,pp 253-268

[3] Kooi,B.W.(2003)'Numerical bifurcation analysis of Ecosystems in spatially homogeneous environment'. Acta Biotheoretica,vol.51,pp189-222

[4] Maria, J.H and Ignacio, B(2003).'Variation in the outcome of population interactions: bifurcations and catastrophes'.Journal of mathematical biology,vol.46,pp571-594

[5] Jia .L, Yican. Z, Zhien,M and James M.H(2004). 'Epidemiological models for mutating pathogen'.Siam Journal of applied mathematics,vol.65 No.1,pp1-23

[6] Baer S.M, Kooi B.W, Kuznetsov Y.A and Thieme H.R (2006).'Multiparametric bifurcation analysis of a basic two stage population model'.Siam J. Appl. Maths,vol.66 No.4,pp1339-1365

[7] Emmanuelle, P and Marino, G. (2000). 'Quantifying the dynamics of prion infection : a bifurcation Analysis of Laurent's model'. J. theoretical.Biology,vol.205, pp283-296

[8] Van Voorn G.A.K, Kooi B.W, and Boer M.P (2010). 'Ecological consequences of global bifurcation in some food chain model' Math Biosci vol. 226 No.2, pp120-133

[9] Ian, M. (1999). 'Steady state bifurcation with Euclidean symmetry'. Journal of American Mathematical Society,vol.351 No.4,pp1575-1603

[10] Tian, M and Shouhong, W.(2004) 'Dynamic bifurcation and stability in the Rayleigh-Benard convection'. Comm. Math. Sci, Vol. 2 No.2,pp159-183

Agwu, I. A, is pursuing a doctoral degree in applied mathematics and a lecturer in mathematics department, Abia State Polytechnics Aba, Nigeria (Corresponding Author: uachilihu@yahoo.com).

Ekaka-a, E.N, is a PhD holder in applied mathematics and a senior Lecturer in the department of mathematics/computer science, Rivers State University of Science & Technology, Port-Harcourt, Nigeria.

Galadima, I.J, is a PhD holder in applied mathematics and a senior Lecturer in the department of mathematics/computer science, IBB University Lapai, Nigeria

Inyama, S.C, is a Professor of Applied mathematics and a lecturer In mathematics department, federal University of Technology Owerri, Nigeria.

Esekhaigbe, E, is a lecturer in mathematics/computer science department, River State University of Science and Technology, Port-Harcourt, Nigeria.