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ON POTENTIAL FLOWS IN THE INTERIOR SCHWARZSCHILD SPACE-TIME

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Abstract— In this paper we consider the interior Schwarzschild solution. Euler and continuity equations have been formulated as potential problem, the equation which represent the fluid flow have been solved by separation. An exact solution is obtained and the solution is exact under a condition involving the density of the hard body, the radial coordinate and the radius of the hard body itself.

Index Terms—Euler and continuity equations, fluid flow, interior Schwarzschild solution, potential problem, Schwarzschild.

1 Introduction

In 1900 Andrew M. Abrahams and Stuart L. Shapiro have presented several new calculations of subsonic relativistic fluid flows in the exterior Schwarzschild metric. The covariant form of the equation of the potential fluid flaw problem for irrotational, isentropic, perfect flow is [1];

$$\psi_{;a}^{;a} + \frac{1-c^2}{c^2}(\ln H)_{,a}\psi^{,a} = 0, \qquad (1.1)$$

where ψ is the potential function of the fluid flow, $H = \frac{h}{h_\infty}$ is the enthalpy scaled to its asymptotic value at infinity, c is the sound speed (the light speed equal 1) and the semicolon and comma stands for the covariant and ordinary derivatives, respectively. The relation between the enthalpy and the fluid flow four-velocity is given by: $Hu_a = \psi_{,a}$. Hence the normalization of the four-velocity yields: $H = (\psi_{,a}\psi^{,a})^{\frac{1}{2}}$.

The interior Schwarzschild space-time metric is defined by [2], [3];

$$ds^2 = \left[\frac{3}{2}\left(1-\frac{r_0^2}{\hat{R}^2}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1-\frac{r^2}{\hat{R}^2}\right)^{\frac{1}{2}}\right]^2 c^2\,dt^2 - \left(1-\frac{r^2}{\hat{R}^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\varphi^2 \qquad (1.2)$$

where, $\hat{R}^2 = \frac{3c^2}{8\pi G\rho}$, G is the gravitational constant, ρ and r_0 is the density and the radius of the hard body respectively.

Besides we put $k_0 = \left(1-\frac{{r_0}^2}{\hat{R}^2}\right)^{\frac{1}{2}}$ and $k = \left(1-\frac{r^2}{\hat{R}^2}\right)^{\frac{1}{2}}$ then, (1.2) becomes,

$$ds^2 = -k^{-2}\,dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\,d\varphi^2 + \left(\frac{3}{2}k_0 - \frac{1}{2}k\right)^2 c^2\,dt^2 \qquad (1.3)$$

The covariant and contravariant metric tensors of (1.3) are,

$$g_{\upsilon\mu} = \begin{pmatrix} -k^{-2} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & c^2\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^2 \end{pmatrix} \qquad (1.4)$$

and

$$g^{\mu\upsilon} = \begin{pmatrix} -k^{2} & 0 & 0 & 0 \\ 0 & -r^{-2} & 0 & 0 \\ 0 & 0 & -r^{-2}\sin^{-2}\theta & 0 \\ 0 & 0 & 0 & c^{-2}\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^{-2} \end{pmatrix} \qquad (1.5)$$

2 Formulation of the Problem

Using the definition of $\psi_{;a}^{;a}$ we obtain [2];

$$\psi_{;a}^{;a} = (-g)^{\frac{-1}{2}}\left[(-g)^{\frac{1}{2}}g^{ab}\psi_{,b}\right]_{,a}. \qquad (2.1)$$

Substituting (1.4) and (1.5) in (2.1) we get;

$$\psi_{;1}^{;1} = -r^{-2}k\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^{-1}\frac{\partial}{\partial r}\left[r^2k\left(\frac{3}{2}k_0-\frac{1}{2}k\right)\frac{\partial\psi}{\partial r}\right], \qquad (2.2)$$

$$\psi_{;2}^{;2} = \frac{-1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\psi}{\partial\theta}\right], \qquad (2.3)$$

$$\psi_{;3}^{;3} = \frac{-1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} \qquad (2.4)$$

and

$$\psi_{;4}^{;4} = c^{-2}\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^{-2}\frac{\partial^2\psi}{\partial t^2}. \qquad (2.5)$$

Assuming axisymmetric $\left(\frac{\partial\psi}{\partial\varphi}=0\right)$ [4]; then involving equations (2.2)-(2.5) in (1.1), yields

$$c^{-2}\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^{-2}\frac{\partial^2\psi}{\partial t^2} - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\psi}{\partial\theta}\right] - \frac{k}{r^2}\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^{-1}\frac{\partial}{\partial r}\left[kr^2\left(\frac{3}{2}k_0-\frac{1}{2}k\right)\frac{\partial\psi}{\partial r}\right] + \frac{1-c^2}{c^2}\left[c^{-2}\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^{-2}\frac{\partial\ln H}{\partial t}\frac{\partial\psi}{\partial t} - k^{-2}\frac{\partial\ln H}{\partial r}\frac{\partial\psi}{\partial r} - r^{-2}\frac{\partial\ln H}{\partial\theta}\frac{\partial\psi}{\partial\theta}\right] = 0 \qquad (2.6)$$

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For incompressible fluid (speed of sound equals speed of light, i.e. $c = 1$), (2.6) becomes [4];

$$c^{-2}\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^{-2}\frac{\partial^2\psi}{\partial t^2}-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\psi}{\partial\theta}\right]-\frac{k}{r^2}\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^{-1}\frac{\partial}{\partial r}\left[kr^2\left(\frac{3}{2}k_0-\frac{1}{2}k\right)\frac{\partial\psi}{\partial r}\right]=0\,. \tag{2.7}$$

3 Analysis

The orthogonality of local coordinate system yields a separable solution;

$$\psi = X(t)\,U(r)V(\theta). \tag{3.1}$$

If we assume zero vorticity and stationary conditions $(\psi_{,t})_{,a}=(\psi_{,a})_{,t}=0$. Then we have,

$$\psi_{,t}=-u_\infty^0=-(1-v_\infty^2)^{-\frac{1}{2}},$$

where v_∞ the magnitude of the asymptotic fluid three-velocity. Hence $\frac{\partial^2\psi}{\partial t^2}=0$. Then, from (3.1) we get,

$$X(t)=c_1t+c_2\,, \tag{3.2}$$

Where $c_1=-u_\infty^0$ and c_2 is an arbitrary constant. Equations (3.1) and (2.7) imply,

$$\frac{U}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right]+kV\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^{-1}\frac{\partial}{\partial r}\left[kr^2\left(\frac{3}{2}k_0-\frac{1}{2}k\right)\frac{\partial U}{\partial r}\right]=0 \tag{3.3}$$

Dividing (3.3) by UV we get,

$$\frac{1}{V\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right]+\frac{k}{U}\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^{-1}\frac{\partial}{\partial r}\left[kr^2\left(\frac{3}{2}k_0-\frac{1}{2}k\right)\frac{\partial U}{\partial r}\right]=0 \tag{3.4}$$

Then we separate the last equation in tofollowing two equations,

$$\frac{d}{d\theta}\left[\sin\theta\frac{dV}{d\theta}\right]=-\beta^2V\sin\theta \tag{3.5}$$

and

$$k\left(\frac{3}{2}k_0-\frac{1}{2}k\right)^{-1}\frac{d}{dr}\left[kr^2\left(\frac{3}{2}k_0-\frac{1}{2}k\right)\frac{dU}{dr}\right]=\beta^2U\,, \tag{3.6}$$

Where, β is an arbitrary constant. A solution of (3.5) is;

$$V(\theta)=\cos\theta, \tag{3.7}$$

where $\beta^2=2$. We write (3.6) in the form,

$$\frac{d}{dr}\left[r^2\sqrt{\hat{R}^2-r^2}\left(3\sqrt{\hat{R}^2-{r_0}^2}-\sqrt{\hat{R}^2-r^2}\right)\frac{dU}{dr}\right]=2\hat{R}^2U\left(\frac{3\sqrt{\hat{R}^2-{r_0}^2}-\sqrt{\hat{R}^2-r^2}}{\sqrt{\hat{R}^2-r^2}}\right). \tag{3.8}$$

If we take, $r=\hat{R}\sin\alpha$ then, (3.8) take the form,

$$\frac{d}{d\alpha}\left[\sin^2\alpha(3k_0-\cos\alpha)\frac{dU}{d\alpha}\right]=2U(3k_0-\cos\alpha) \tag{3.9}$$

Or,

$$\frac{d^2U}{d\alpha^2}+\left[\frac{2\cos\alpha}{\sin\alpha}+\frac{\sin\alpha}{(3k_0-\cos\alpha)}\right]\frac{dU}{d\alpha}-\frac{2U}{\sin^2\alpha}=0 \tag{3.10}$$

Substituting in (3.10) with,

$$U=\frac{W}{\sin\alpha\sqrt{(3k_0-\cos\alpha)}}. \tag{3.11}$$

Then, equation (3.10) transformed to the standard form [5];

$$\frac{d^2W}{d\alpha^2}+L(\alpha)W=0 \tag{3.12}$$

Where,

$$L(\alpha)=-\frac{2}{\sin^2\alpha}-\frac{1}{2}\frac{d}{d\alpha}\left[\frac{2\cos\alpha}{\sin\alpha}+\frac{\sin\alpha}{(3k_0-\cos\alpha)}\right]-\frac{1}{4}\left[\frac{2\cos\alpha}{\sin\alpha}+\frac{\sin\alpha}{(3k_0-\cos\alpha)}\right]^2,$$

Or,

$$L(\alpha)=-\frac{1+\cos^2\alpha}{\sin^2\alpha}-\frac{\cos^2\alpha+6k_0\cos\alpha-2}{4(3k_0-\cos\alpha)^2}+\frac{\cos\alpha}{(3k_0-\cos\alpha)}.$$

A particular solution of equation (3.12) is;

$$W=4\sin^2\alpha\,(3k_0-\cos\alpha)^2, \tag{3.13}$$

under the condition

$$\left(130\cos^2\alpha-396k_0\cos\alpha+216{k_0}^2-20\right)\sin^2\alpha=0$$

or

$$\left(130k^2-396k_0k+216{k_0}^2-20\right)\frac{r^2}{\hat{R}^2}=0 \tag{3.14}$$

Substituting (3.13) in (3.11) we get,

$$U=4\sin\alpha\,(3k_0-\cos\alpha)^{\frac{3}{2}} \tag{3.15}$$

or

$$U=\frac{4r}{\hat{R}^2}\left(3\sqrt{\hat{R}^2-{r_0}^2}-\sqrt{\hat{R}^2-r^2}\right)^{\frac{3}{2}} \tag{3.16}$$

Substituting (3.2) , (3.7) and (3.16) in(3.1), we get

$$\psi=(-u_\infty^0\,t+c_2)\cos\theta\left[\frac{4r}{\hat{R}^2}\left(3\sqrt{\hat{R}^2-{r_0}^2}-\sqrt{\hat{R}^2-r^2}\right)^{\frac{3}{2}}\right]. \tag{3.17}$$

Equations (3-17) represent an exact solution of (2-7) under the condition (3-14).

4 Conclusion

Under condition (3-14), the flow function ψ given by (3.17) gives a solution of equation (2.7) for a high density of a hard body . For example if we consider a hard body of radius equal to the average radius of the earth, then the flow function ψ available only for the hard body's density of range between 3.5×10^{12} and 5.6×10^{12} density unit.

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