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Quantifying Some Simple Chaotic Models Using Lyapunov Exponents

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Abstract: Lyapunov exponents are used to quantify various dynamical systems which have been found to display chaotic behaviour by embedding them in a common structural framework.

Index Terms: Chaotic behaviour, non-linear dynamical systems, Lyapunov exponents, strange attractors

1 Introduction

Chaos theory looks at the study of deterministic dynamical systems that are very sensitive to initial conditions. Small differences in initial conditions can lead to widely diverging outcomes, for such systems making long term predictions generally becomes impossible. Chaotic phenomena have been observed in numerous systems in the science and engineering fields [1]. Potential applications in engineering fields to the study and control of chaotic systems has become important, specifically chaos control and synchronization [2]. Several examples of a simple feedback system, comprising a single linear dynamic element and a static nonlinear function have shown to display chaotic behaviour [3], [4], [5]. To simplify the analysis, the nonlinearity is taken to be a piecewise-linear function as in Ogorzalek [3], Brockett [4] and Chua et al [5] all with third-order dynamics. Unification of these systems in to a model where the linear element has a general third-order transfer function and the nonlinearity is made up of up to five linear segments has been done [6]. In this paper the above model's chaotic behaviour are quantified using Lyaponov exponents.

2 Background

2.1 General Model Description

Considering a description to represent a general third-order system with a continuous piecewise nonlinearity of up to five linear segments.

The feedback systems can be represented as in Fig 1.

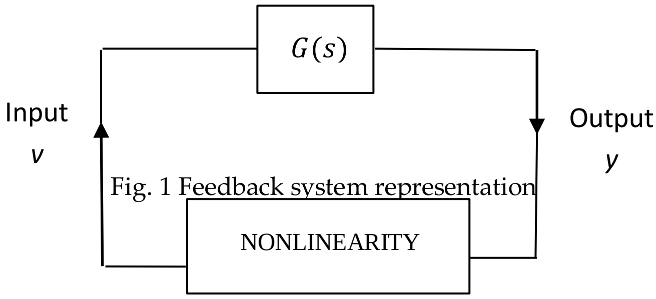

Fig. 1 Feedback system representation

where the linear element has the transfer function

$$G(s) = \frac{a_1 s^2 + a_2 s + a_3}{s^3 + b_1 s^2 + b_2 s + b_3} \qquad (1)$$

and the nonlinear element has the following piecewise-linear characteristic as shown in Fig 2.

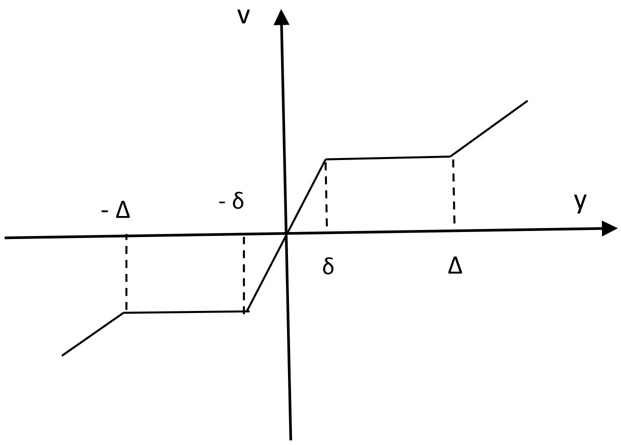

Fig. 2 Piecewise nonlinearity

This is represented analytically as follows:

$$v = f(y) = \begin{cases} y/\delta, & |y| < \delta \\ \mathrm{sgn}(y), & \delta < |y| < \Delta \end{cases} \qquad (2)$$

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$$My - (M\Delta - 1)\operatorname{sgn}(y)\ ,\ |y| > \Delta$$

2.2 Lyapunov Exponents

Lyapunov exponents are the average exponential rates of divergence or convergence of nearby orbits in the phase space and are clearly fundamental importance in studying chaotic behaviour. Positive Lyapunov exponents indicate orbital divergence and chaos and the magnitude of the exponent reflects the time scale on which the system dynamics become unpredictable. Negative Lyapunov exponents set the time scale on which transients or perturbations of the system state will decay.

The spectrum of Lyapunov exponents are defined in a manner that is particularly useful for the computational algorithm used to calculate them. Given a continuous dynamical system in an n-dimensional phase space, the long term behaviour of an infinitesimal n-sphere of initial conditions is monitored. The sphere will become an n-ellipsoid due to the locally deforming nature of the flow. The i^{th} one-dimensional Lyapunov exponent is then defined in terms of the length of the ellipsoidal principal axis $P_i(t)$ [7].

$$\lambda_i \triangleq \lim_{t\to\infty} \frac{1}{t} \log_2 \left(\frac{P_i(t)}{P_i(0)} \right) \qquad (3)$$

where the λ_i are ordered from largest to smallest.

Thus, the Lyapunov exponents are related to the expanding or contracting nature of different directions in phase space. Since the orientation of the ellipsoid changes continuously as it evolves, the directions associated with a given exponent vary in a complicated way through the attractor. Rearranging (1) we can see that $P_i(t)$ is

$$P_i(t) = P_i(0)\, 2^{\lambda_i t} \qquad (4)$$

for sufficiently large t. So the linear extent of the ellipsoid grows as $2^{\lambda_1 t}$, the area defined by the first two principal axes grows as $2^{(\lambda_1 + \lambda_2)t}$, the volume defined by the first three principal axes grows as $2^{(\lambda_1 + \lambda_2 + \lambda_3)t}$ and so on.

Axes that are on the average expanding correspond to positive Lyapunov exponents and those that are contracting correspond to negative Lyapunov exponents. The exponential expansion indicated by a positive Lyapunov exponent is incompatible with motion on a bounded attractor unless some sort of folding process merges wildly separated trajectories. Each positive exponent reflects a 'direction' in which the system experiences the repeated stretching and folding that de-correlates nearby states on the attractor. Therefore the long-term behaviour of an initial condition that is specified with any uncertainty cannot be predicted, this is termed chaos. An attractor for a dissipative system with one or more positive Lyapunov exponents is said to be 'strange' or 'chaotic'.

The magnitudes of the Lyapunov exponents quantify an attractor's dynamics in information theoretic terms. The exponents measure the rate at which system processes create or destroy information. Thus the exponents are expressed in bits of information per second or bits/orbit for a continuous system and bits/iteration for a discrete system. So for example an attractor with a positive exponent of magnitude 2.16 bits/second and if the initial point were specified with an accuracy of one part million, i.e. 20 bits, the future behaviour could not be predicted after about 9 seconds. All strange attractors in a three-dimensional phase space have the same spectral type (+ , 0 , -). A positive exponent indicating chaos within the attractor, a zero exponent for the slower than exponential motion along an orbit and a negative exponent so that the phase space contains an attractor.

2.3 Computation of Lyapunov Exponents

The technique for determining the complete Lyapunov spectrum from a set of differential equations has been developed [8], [9]. Lyapunov exponents are defined as the long-term evolution of the axes of an infinitesimal sphere of states. This procedure could be implemented by defining the principal axes with initial conditions whose separations are as small as computer limitations allow and evolving these with the nonlinear equations of motion. One problem with this approach is that in a chaotic system we cannot guarantee the condition of small separations for times on the order of hundreds of orbital periods for the convergence of the spectrum.

This problem can be avoided with the use of a phase space plus tangent space approach. A 'fiducial' trajectory (the centre of the sphere) is defined by the action of the nonlinear equations of motion on some initial condition. Trajectories of points on the surface of the sphere are defined by the action of the linearized equations of motion on points infinitesimally separated from the fiducial trajectory. In particular, the principal axes are defined by the evolution via the linearized equations of an initially orthonormal vector frame anchored to the fiducial trajectory. This procedure is implemented by creating the fiducial trajectory by way of integrating the nonlinear equations for some post-transients initial condition. Simultaneously, the linearized equations of motion are integrated for n different initial conditions defining an arbitrarily oriented frame of n orthonormal vectors. As well as each vector diverging in magnitude, there appears another problem of a singularity occurring. In chaotic systems, each vector tends to fall along the local direction of the most rapid growth. To overcome these two problems a technique from linear algebra is repeatedly used known as the Gram-Schmidt reorthonormalization (GSR) procedure on the vector frame. The importance of the orientation-preserving property of GSR is seen from the alternative definition of the spectrum of Lyapunov exponents, where the rate of length growth determines λ_1, the rate of area growth determines $\lambda_1 + \lambda_2$ and in general the rate of k-volume growth determines $\sum_1^k \lambda_i$.

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When GSR is used the initial and final volumes of the elements of each dimension:

$$[\,L(t_j), L'(t_{j+1}); A(t_j), A'(t_{j+1}); V(t_j), V'(t_{j+1}) \ldots]$$

are recorded and used to update the running exponential growth rates. If m replacement elements spanning a time t have been used, the exponential growth rate of the first principal axis is given by

$$(\lambda_1)_m = \frac{1}{t} \sum_{j=1}^{m} log_2 \left[\frac{L'(t_{j+1})}{L(t_j)}\right] \qquad (5)$$

which is identical to the growth rate,

$$\lambda_1 = \frac{1}{t}\ log_2 \left[\frac{L'(t_1)}{L(t_0)}\right] \qquad (6)$$

that would have been obtained from the evolution of a single length element had we been able to follow it for time t. Similarly,

$$(\lambda_1 + \lambda_2)_m = \frac{1}{t} \sum_{j=1}^{m} log_2 \left[\frac{A'(t_{j+1})}{A(t_j)}\right] \qquad (7)$$

which is identical to the exponential growth rate of the area element defined by the first two principal axis vectors had we been able to follow it for a long time t. The remaining exponents are defined in a similar way.

3. Results

The Lyapunov exponents for the different dynamical systems presented in section 1 are calculated. The computer programme used to calculate the Lyapunov exponents is a modified version to the one given by Wolf et al [10]. The dynamical equations of motion of the different systems together with their linearized equations of motion are used in the programme.

The systems whose Lyapunov exponents are to be calculated are given below with their transfer functions and associated pole distributions for their observed complex behaviour in Table 1.

Model	Transfer Function	Poles
Chua's circuit	$G(s) = \dfrac{2.7\delta(s^2 + 0.7s + 7.2)}{s^3 + 2.5s^2 + 4s + 12.9}$	$s \approx -2.8$ $s \approx 0.1 \pm j\,2.2$
Brockett's system	$G(s) = \dfrac{5.4}{s^3 + s^2 + 1.25s + 3.6}$	$s \approx -1.6$ $s \approx 0.3 \pm j\,1.44$
Ogorzalek's system	$G(s) = \dfrac{40}{s^3 + 5s^2 + 6s + 36}$	$s \approx -5.2$ $s \approx 0.09 \pm j\,2.6$

Table 1

The corresponding calculated Lyapunov exponents are given in Table 2.

Model	Lyapunov Exponents
Chua's Circuit (Chaotic)	(0.36, -0.003, -2.46)
Brockett's System (Chaotic)	(0.14, -0.008, -1.57)
Ogorzalek's System (Chaotic)	(0.13, -0.006, -7.34)
Ogorzalek's System (Stable)	(-0.13, -2.54, -4.54)

Table 2

7. Conclusions

It is found that the spectrum of Lyapunov exponents for our different models, Chua, Brockett and Ogorzalek correspond closely to that describing a strange attractor (+, 0, -). We see in Table 2 above that in each case, for certain parameter values, we have one positive, one approximately zero and one negative exponent signifying that the dynamical structure of the different systems is indeed one of chaotic behaviour. The different dynamical behaviour of the models depends upon the different parameter values chosen. In our systems if the appropriate parameter values are chosen such that the systems are structurally stable then we see from Table 2, for Ogorzalek's system (stable) that the corresponding Lyapunov spectrum is that of three negative exponents indicating a stable fixed point as expected. Knowing the dynamical behaviour of a system for different parameter values is important from a control perspective and it allows one to avoid undesirable types of system behaviour.

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