Special Family of Ruled Surfaces in Euclidean 3-Space

Soukaina Ouarab, Amina Ouazzani Chahdi

Abstract—The purpose of the present paper is to construct a family of ruled surfaces with Frenet frame of an arbitrary non-cylindrical and regular ruled surface in Euclidean 3-space. We study the most important characteristic properties of that special family of ruled surfaces such as the Gaussian curvature, the mean curvature, the striction curve and give their associated characterizations. Moreover, we apply our study for special regular and non-cylindrical ruled surfaces whose striction curve is the unit circle, the general circular helix and the general non-circular helix, respectively.

Index Terms—Euclidean 3-space, Frenet frame, geodesic curvature, geodesic torsion, Gaussian curvature, mean curvature, normal curvature, ruled surface, striction curve.

1 INTRODUCTION

In the classical differential geometry, the study of special surfaces and their properties is one of the most principal aims. The well-known types of special surfaces are helicoid, rotational, cyclic, canal, revolution, ruled surfaces...and are useful in many domains of applications, such as mathematical physics, moving geometry, kinematics and the domain of Computer Aided Geometric Design (CAGD).

Ruled surface [9] is one of the most special and fascinating surfaces which is defined by the moving of a straight line (ruling) around a curve (base curve). An important number of studies of ruled surfaces has been realized with special frames. Effectively, in [5], the authors have studied ruled surface with Frenet frame of a regular curve, they have investigated its properties and gave examples of ruled surfaces with Frenet frame of general and slant helices in euclidean 3-space. A few years ago, in [2,4], the authors have studied ruled surface with Darboux frame in euclidean 3-space, but in [6] it have been studied in Mikonwski 3-space. Recently, in [1] ruled surface and its properties have been studied with Alternative moving frame in euclidean 3-space where an important part has been investigated for examples of ruled surfaces with alternative moving frame of general and slant helices. Frenet frame of ruled surface [8], is one of the remarkable frames which has interested many researchers by the properties that it is defined along the striction curve of a non-cylindrical ruled surface and it have been studied with different methods and with which many important new definitions and characterizations of ruled surfaces have been investigated such as slant ruled surface [3, 7].

In this work, we are interested in Frenet frame of ruled surface. Therefore, we construct and study special family of ruled surfaces with Frenet frame and which is defined precisely by a lineal combination of Frenet frame vectors of an arbitrary non-cylindrical ruled surface in euclidean 3-space. Moreover, we present a part of applications of our study, in which we give examples with illustrations in the real Euclidean 3-space $\mathbb{R}^3$ of special ruled surfaces whose striction curve is the unit circle, the general circular helix and the general non-circular helix.

2 PRELIMINARIES

Let be $I$ and $\mathbb{E}^3$ an open interval of $\mathbb{R}$ and an euclidean 3-space, respectively. A ruled surface $\Phi$ is a special surface generated by a continuous moving of a line along a curve and has the parametrization

$$\Phi(s, v) = c(s) + vq(s),$$

where the curve $c = c(s)$ is called base curve of ruled surface and $q = q(s)$ is a unit direction vector of an oriented line in $\mathbb{E}^3$ whose various positions are called rulings.

The unit normal vector denoted by $m = m(s, v)$ on ruled surface $\Phi$ at a regular point $\Phi(s, v)$ is given by

$$m(s, v) = (\Phi_x \times \Phi_v)/\|\Phi_x \times \Phi_v\|,$$

(1)

Where $\Phi_x = \partial \Phi/\partial s$ and $\Phi_v = \partial \Phi/\partial v$.

The first $I$ and the second II fundamental forms of ruled surface $\Phi$ at a regular point $\Phi(s, v)$ are defined respectively by

$$I(\Phi_x ds + \Phi_v dv) = E ds^2 + 2Fdsv + Gdv^2,$$

$$II(\Phi_x ds + \Phi_v dv) = eds^2 + 2fdsv + gdv^2,$$

Where $E = \|\Phi_x\|^2, F = \langle \Phi_x, \Phi_v \rangle, G = \|\Phi_v\|^2$, and $e = \langle \Phi_x, m \rangle, f = \langle \Phi_v, m \rangle, g = \langle \Phi_v, m \rangle = 0$.

The Gaussian curvature $K$ and the mean curvature $H$ of $\Phi$ at a regular point $\Phi(s, v)$ are given respectively by:

$$K = -f^2/(EG - F^2), \quad H = (Ge - 2Ff)/(2(EG - F^2)).$$

Definition: A regular surface is developable (resp. minimal) if its Gaussian curvature (resp. mean curvature) vanishes identically.

The normal curvature $\rho_n$ the geodesic curvature $\rho_g$ and the geodesic torsion $\tau$ of a regular curve on a regular surface are defined respectively by:
\[ \rho_n = \langle n, T^t \rangle, \rho_s = \langle n \times T, T^t \rangle, \theta_s = -\langle n \times T, n^t \rangle, \]

where \(\overrightarrow{n}\) and \(T\) are the unit normal of surface along the curve \(c = c(s)\) and the unit tangent of the curve \(c = c(s)\), respectively.

**Definition:** A regular curve on a regular surface is an asymptotic line (resp. geodesic curve, resp. principal line) if its normal curvature (resp. geodesic curvature, resp. geodesic torsion) vanishes identically.

Supposing that \(\phi\) is a regular and non-cylindrical ruled surface with \(q(s) = 1\). If \(v\) infinitely decreases, then along a ruling \(s = s_0\), the unit normal \(m\) approaches a limiting direction. This direction is called the asymptotic normal (central tangent) direction denoted by \(a(s_0)\) and from (1) \(a(s) = \lim_{h \to \infty} m(s_0, y) = (q(s_0) \times q'(s_0))/\|q'(s_0)\|\).

The point at which the unit normal of \(\phi\) is perpendicular to \(a\) is called the striction point (or central point) \(C\) and the set of striction points of all rulings is called striction curve of the ruled surface which we can denote by \(\beta = \beta(s)\) and it is defined by

\[ \beta(s) = c(s) - \left(\frac{c'(s), q'(s)}{\|q'(s)\|^2}\right) q(s). \]

The vector \(h\) defined by \(h = a \times q\) is called central normal vector. Then the orthonormal system \(\{C,q,h,a\}\) is called Frenet frame of the ruled surface \(\phi\) where \(C\) is the central point and \(q,h,a\) are unit vector of ruling, central normal and central tangent, respectively. The set of all bound vectors \(q(s)\) at origin \(O\) constitutes a cone which is called directing cone of ruled surface \(\phi\).

The end points of these unit vectors circumscribe a spherical curve \(k_1\), called the spherical image of the ruled surface whose arc is denoted by \(s_1\). The derivative formulae of Frenet frame of ruled surface \(\phi\) with respect to the arc \(s\) are given by

\[ q' = k_1 h, \quad h' = -k_1 q + k_2 a, \quad a' = -k_2 h, \]

where \(k_1 = ds_1/l ds, \quad k_2 = ds_3/l ds\), \(s_1, s_3\) are the arcs of the spherical curves \(k_1, k_2\) circumscribed by the bound vectors \(q(s)\) and \(a(s)\).

3 **FAMILY OF Ruled SURFACES WITH Frenet of an ARBITRARY NON-CYLINDRICAL RULEd SURF ace IN EUCLIDEAN 3-SPACE**

\[ C^k(k \geq 3) \text{ in the euclidean 3-space } E^3 \text{ defined by} \]

\[ \phi:(s,v) \in I \times R \mapsto \beta(s) + v q(s), \quad \|q(s)\| = 1, \quad (2) \]

where \(\beta = \beta(s)\) and \(q = q(s)\) are a normal parametrization of its striction curve and the directing vectors of its rulings, respectively.

Considering the family of ruled surfaces defined in \(E^3\) by striction curve and Frenet frame vectors of ruled surface \(\phi\) defined in (1), as follows

\[ \Psi(s,v) = \beta(s) + v(x_1 q(s) + x_2 h(s) + x_3 a(s)), \quad (3) \]

Where \(x_1, x_2, x_3\) are three constants satisfying \(x_1^2 + x_2^2 + x_3^2 = 1\).

\[ \beta = \beta(s) \text{ is the striction curve of } \phi, \text{ it verifies } \beta', \beta'' \geq 0, \text{ then } < \beta', h > = 0, \text{ which means that } \beta' \text{ belongs to the plan generated by both vectors } q \text{ and } a. \]

Therefore, we can write

\[ \beta' = a_1 q + a_2 a, \]

Where \(a_1: s \in I \mapsto a_1(s)\) and \(a_2: s \in I \mapsto a_2(s)\) are two real functions satisfying \(a_1 + a_2 = 1\).

Differentiating (3) with respect to \(s\) and \(v\), respectively, we get

\[ \Psi'_s = (a_1 - v x_1 k_1) q + v (x_1 k_1 - x_2 k_2) h + (a_2 + v x_2 k_2) a, \quad (4) \]

\[ \Psi'_v = x_1 q + x_2 h + x_3 a, \quad (5) \]

then, the unit normal denoted by \(n = n(s,v)\) on the family of ruled surfaces defined in (3) at a regular point \(\Psi(s,v)\) is given by

\[ n = n(s,v) = (\Psi'_s \times \Psi'_v) / \| \Psi'_s \times \Psi'_v \| = X / \|X\|, \]

Where

\[ X = (-x_2 a_2 + v A) q + (x_1 a_2 - x_3 a_1 + v B) h + (x_2 a_1 - x_3 c) a, \]

\[ \|X\| = \sqrt{(-x_2 a_2 + v A)^2 + (x_2 a_1 - x_3 a_1 + v B)^2 + (x_2 a_1 - x_3 a_1)^2}, \]

where \(A = x_3 x_1 k_1 - (x_1^2 + x_2^2) k_2, \quad B = x_2 (x_1 k_1 + x_3 k_1), \quad C = (x_1^2 + x_2^2) k_1 - x_1 x_2 k_2.\)

From (4) and (5), the components \(E, F\) and \(G\) of the first fundamental form of \(\Psi\) at a regular point \(\Psi(s,v)\) are given by:

\[ E = 1 + 2 v x_2 (a_2 k_2 - a_1 k_1) + v^2 [(x_1 k_1 - x_3 k_2)^2], \quad F = x_1 a_1 + x_3 a_2, \quad G = 1. \]

Differentiating \(\Psi'_s\) and \(\Psi'_v\) in (4) and (5) with respect to \(s\) and \(v\), we get

\[ \Psi''_{ss} = [a_1' - v (x_2 k_2 + k_1 (x_1 k_1 - x_3 k_2)) a + (k_1 a_1 - k_2 a_2 + v (x_1 k_1 - x_3 k_2 - x_2 k_2))(x_1 k_1 - x_3 k_2)] h \]

Let be \(\phi\) a non-cylindrical and regular ruled surface of class
Then, from these last and (1), the components $e$, $f$, and $g$ of the second fundamental form of $\Psi$ at a regular point $\Psi(s,0)$ along its base curve $\beta = \beta(s)$ are given as follows:

\[
e(s,0)=\frac{\left[x_1'(a_2-a_1)+\alpha(a_2-a_1)\right]}{R},
\]

\[
f(s,0)=\frac{\left[x_2'(a_2k_1+a_2k_2)+(x_1k_1-x_2k_2)(x_1a_2-x_2a_1)\right]}{R},
\]

\[
g(s,0)=0.
\]

Where $R=\sqrt{x_1'^2+(x_1a_2-x_2a_1)^2}$.

Consequently, from these last and the components of the first fundamental form, the Gaussian curvature $K$ and the mean curvature $H$ of the family of ruled surfaces defined in (3) at a regular point $\Psi(s,0)$ of the base curve $\beta = \beta(s)$ are investigated as follows:

\[
K(s,0)=-\frac{x_1'^2(k_1-k_2)(x_1a_2-x_2a_1)}{R^2}+\frac{(x_1k_1-x_2k_2)(x_1a_2-x_2a_1)}{R},
\]

\[
H(s,0)=\frac{x_2'(a_2k_1+a_2k_2)+(x_1k_1-x_2k_2)(x_1a_2-x_2a_1)}{2R}.
\]

Corollary:

$\Psi$ is developable along the base curve $\beta$ if and only if the curvatures $k_1$ and $k_2$ satisfy the equation

\[
x_1(x_2k_1+a_2k_2)+(x_1k_1-x_2k_2)(x_1a_2-x_2a_1)=0.
\]

Corollary:

$\Psi$ is minimal along the base curve $\beta$ if and only if the curvatures $k_1$ and $k_2$ satisfy the equation

\[
x_1(x_2(a_2k_1+a_2k_2))+(x_1k_1-x_2k_2)(x_1a_2-x_2a_1)
\]

\[
-2x_1^2(x_1a_1+k_1a_1+k_2)
\]

\[
-2x_1^2(x_1a_1+k_1a_1+k_2)(x_1a_2-x_2a_1)=0.
\]

Another hand, denoting by $n_0$ and $T$ the unit normal on $\Psi$ at a regular point $\Psi(s,0)=\beta(s)$ and the unit tangent of $\beta = \beta(s)$ at the point $\beta(s)$. We have

\[
n_0=n(s,0)=[-x_2a_2q+(x_1a_2-x_2a_1)h+x_2a_aa]/R,
\]

\[
T=\alpha_1q+\alpha_2a.
\]

From these last we get

\[
n_0T=\alpha_2(x_2a_1-x_2a_1)q+x_1h-a_2(x_1a_2-x_2a_1)\sqrt{x_2^2+(x_1a_2-x_2a_1)^2},(6)
\]

\[
T'=\alpha_1'q+(\alpha_2k_1-\alpha_2k_2)h+\alpha_2'a,
\]

\[
n_0'=(Lq+Mh+Na)/(x_2^2+(x_1a_2-x_2a_1)^2)^{3/2},(8)
\]

\[
L=-[x_2a_2'+k_1(x_1a_2-x_2a_1)](x_2^2+(x_1a_2-x_2a_1)^2)
\]

\[
+2x_2a_2'k_1(x_1a_2-x_2a_1)(x_1a_2-x_2a_1),
\]

\[
M=-[x_2a_2'+k_2(x_1a_2-x_2a_1)](x_2^2+(x_1a_2-x_2a_1)^2)
\]

\[
-(x_2a_2'-x_1a_1)(x_1a_2-x_2a_1)^2,
\]

\[
N=[k_1(x_1a_2-x_2a_1)+x_2a_1'](x_2^2+(x_1a_2-x_2a_1)^2)
\]

\[
-x_2a_2'(x_2a_2'-x_1a_1)(x_1a_2-x_2a_1).
\]

Consequently, from (6), (7) et (8), the normal curvature $\rho_n$, the geodesic curvature $\rho_g$, and the geodesic torsion $\theta_g$ of $\beta = \beta(s)$ on ruled surfaces defined in (3) are investigated, respectively as follows:

\[
\rho_n=[x_2(x_1a_2-x_2a_1)q+(x_1a_2-x_2a_1)k_1-k_2]/R,
\]

\[
\rho_g=[x_1a_1'+x_2a_2+(x_2a_2'-x_1a_1)(k_1-k_2)]/R,
\]

\[
\theta_g=-[(x_1a_2-x_2a_1)L+x_2M-k_1(x_2a_2-x_2a_1)N]/R^4.
\]

Where

Corollary:

$\beta = \beta(s)$ is an asymptotic line for $\Psi$ if and only if $x_1(a_2a_2'-a_1a_1')+(x_1a_2-x_2a_2)(a_1k_1-a_2k_2)=0$.

Corollary:

$\beta = \beta(s)$ is a geodesic curve for $\Psi$ if and only if $x_1a_1'+x_2a_2+(x_2a_2'-x_1a_1)(k_1-k_2)=0$.

Corollary:

$\beta = \beta(s)$ is a principal line for $\Psi$ if and only if $x_1(x_2a_2-x_2a_1)L+x_2M-k_1(x_2a_2-x_2a_1)N=0$.

Another hand, when $\Psi$ is non-cylindrical,i.e., $x_1^2(k_1^2+k_2^2)+(x_1k_1-x_2k_2)^2 \neq 0$, its striction curve denoted by $\mu = \mu(s)$ is obtained as follows:

\[
\mu=\frac{-x_2(-\alpha_1k_1+\alpha_2k_2)}{Sx_2q+x_2h+x_2a},
\]

Where $S=x_1^2(k_1^2+k_2^2)+(x_1k_1-x_2k_2)$.

Which implies the following corollaries:

Corollary:

$\beta$ is the striction curve of $\Psi$ if and only if $-\alpha_1k_1+\alpha_2k_2=0$.

Corollary:

$\beta$ is the striction curve of the family of ruled surfaces defined by $(x,v)\in I \times R \mapsto \beta(s)+v(x_2q+x_2a)$, where $x_1^2+x_2^2=1$.

4. Applications
In the following part, we present study of some examples of the family of ruled surfaces which is defined in (3). We investigate its properties and give illustrations of some special ruled surfaces in the family studied using the MATLAB software. The first, the second and the third example are given with non-cylindrical and regular ruled surface whose striction curve is the unit circle, the circular helix and the general non-circular helix, respectively.

Example 1:

Considering the non-cylindrical and regular ruled surface \( \vec{\phi} = (\phi_1, \phi_2, \phi_3) \) defined in \( \mathbb{R}^3 \) by the unit circle \( (0, \sin s, \cos s) \) as its striction curve, as follows:

\[
\begin{align*}
\phi_1 &= \cos s + (\sqrt{2}/2) \sin s, \\
\phi_2 &= \sin s - (\sqrt{2}/2) \cos s, \\
\phi_3 &= \sqrt{2}.
\end{align*}
\]

Its Frenet frame \( (\beta(s), q(s), h(s), a(s)) \) the associated curvatures \( k_1 \) and \( k_2 \) and the functions \( \alpha_1 \) and \( \alpha_2 \) are given, respectively by:

\[
q = (1/\sqrt{2})(\sin s, -\cos s, 1),
\]
\[
h = (\cos s, \sin s s, 0),
\]
\[
a = (1/\sqrt{2})(\sin s s, \cos s, 1),
\]
\[
k_1 = k_2 = 1/\sqrt{2}, \quad \alpha_1 = -1/\sqrt{2}, \quad \alpha_2 = 1/\sqrt{2}.
\]

Hence, the special family studied \( \Psi \) of ruled surfaces which is constructed with Frenet frame of \( \phi \) and its characteristic properties are given respectively by

\[
\Psi = (\psi_1, \psi_2, \psi_3),
\]

Where

\[
\begin{align*}
\psi_1 &= (1 + x v) \cos s, \\
\psi_2 &= (1 + \nu(x - x_0)/\sqrt{2}) \sin s, \\
\psi_3 &= \nu(x_0 + x_1)/\sqrt{2}.
\end{align*}
\]

\[
\begin{align*}
K(s, 0) &= -((x_0^2 - x_0^2) \nu^2/(2x_0^2 + (x_1 + x_2)^2), \\
H(s, 0) &= (x_0 + x_1)((x_0 - x_0)^2 - 1)/(2x_0^2 + (x_1 + x_2)^2), \\
\rho_n &= -(x_0 + x_1)/\sqrt{2x_0^2 + (x_1 + x_2)^2}, \\
\rho_s &= -2x_0/\sqrt{2x_0^2 + (x_1 + x_2)^2}, \\
\theta_s &= 0, \\
\mu &= (\mu_1, \mu_2, \mu_3).
\end{align*}
\]

Where

\[
\begin{align*}
\mu_1 &= \cos s - 2\nu(x_0(x_0 - x_0)/(2x_0^2 + (x_1 - x_0)^2))\sin s/s + x_2 \cos s, \\
\mu_2 &= \sin s + 2\nu(x_0(x_0 - x_0)/(2x_0^2 + (x_1 - x_0)^2))\sin s/s + x_2 \sin s, \\
\mu_3 &= -\sqrt{2}x_0 x_0(x_0 + x_0)/(2x_0^2 + (x_1 - x_0)^2).
\end{align*}
\]

In the following, some illustrations of some special ruled surfaces which are defined in the family (9)
These above figures are plotted with $(s,v) \in [-2\pi,2\pi] \times [-4,4]$.

Example 2:

Considering the non-cylindrical and regular ruled surface $\phi = (\phi_1, \phi_2, \phi_3)$ in $\mathbb{R}^3$ defined with the general helix

\[
2\beta(s) = (1/2)(\sin(\sqrt{2}s), -\cos(\sqrt{2}s), \sqrt{2}s),
\]

as its striction curve as follows:

\[
2\phi_1 = (1/2)\sin(\sqrt{2}s) - v\cos(\sqrt{2}s)/\sqrt{2},
\]

\[
2\phi_2 = -(1/2)\cos(\sqrt{2}s) - v\sin(\sqrt{2}s)/\sqrt{2},
\]

\[
2\phi_3 = (s+v)/\sqrt{2}.
\]

Its Frenet frame $\{\Psi_1, \Psi_2, \Psi_3\}$, the associated curvatures $k_1, k_2$ and the functions $\alpha_1, \alpha_2$ are given, respectively by

\[
2q = 1/\sqrt{2}(-\cos s, -\sin s, 1),
\]

\[
h = (\sin(\sqrt{2}s), -\cos(\sqrt{2}s)) / \sqrt{2},
\]

\[
a = 1/\sqrt{2}(\cos(\sqrt{2}s), \sin(\sqrt{2}s), 1),
\]

\[
k_1 = k_2 = 1, \quad \alpha_1 = 0, \quad \alpha_2 = 1.
\]

Hence, the studied family of ruled surfaces, which is constructed with Frenet frame of $\phi$ and its characteristic properties are investigated, respectively by

\[
2\Psi = (2\Psi_1, 2\Psi_2, 2\Psi_3), \quad (10)
\]

Where

\[
2\Psi_1 = (1/2)\sin(\sqrt{2}s) + v[(-x_1 + x_3)\cos(\sqrt{2}s)/\sqrt{2} + x_2\sin(\sqrt{2}s)],
\]

\[
2\Psi_2 = -(1/2)\cos(\sqrt{2}s) + v[(-x_1 + x_3)\sin(\sqrt{2}s)/\sqrt{2} - x_2\cos(\sqrt{2}s)],
\]

\[
2\Psi_3 = (s/\sqrt{2}) + v(x_1 + x_3)/\sqrt{2}.
\]

\[
2K(s,0) = -[(x_2^2 + x_3^2 - x_3x_3)/(x_2^2 + x_3^2)],
\]

\[
2H(s,0) = -[(x_1 + 2x_3(x_2^2 + x_3^2 - x_3x_3))/2(x_3^2 + x_3^2)^{3/2}],
\]

\[
2\rho = -x_1/\sqrt{x_2^2 + x_3^2},
\]

\[
\rho_g = -x_2/\sqrt{x_2^2 + x_3^2},
\]

\[
3\theta = 1,
\]

\[
2\mu = (2\mu_1, 2\mu_2, 2\mu_3)
\]

Where

\[
2\mu_1 = 2\beta_1 - x_2[(-x_1 + x_3)\cos(\sqrt{2}s)/\sqrt{2} + x_2\sin(\sqrt{2}s)][2x_2^2 + (x_1 - x_3)^2],
\]

\[
2\mu_2 = 2\beta_2 - x_2[(-x_1 + x_3)\sin(\sqrt{2}s)/\sqrt{2} - x_2\cos(\sqrt{2}s)][2x_2^2 + (x_1 - x_3)^2],
\]

\[
2\mu_3 = 2\beta_3 - x_2(x_1 + x_3)[2\sqrt{2}x_2^2 + \sqrt{2}(x_1 - x_3)^2].
\]

In the following, some illustrations of some special ruled surfaces which are defined in the family (10)
These above figures are plotted with \((x, y) \in \Pi \times [4, 4]\), and the following values of the constants:

- Fig. 7. Ruled surface \(2\Psi \) with \(x_1 = 1, x_2 = x_3 = 0\).
- Fig. 10. Ruled surface \(2\Psi \) with \(x_1 = x_3 = 1/\sqrt{2}, x_2 = 0\).
- Fig. 8. Ruled surface \(2\Psi \) with \(x_2 = 1, x_1 = x_3 = 0\).
- Fig. 11. Ruled surface \(2\Psi \) with \(x_1 = x_2 = x_3 = 1/\sqrt{3}\).
- Fig. 9. Ruled surface \(2\Psi \) with \(x_1 = x_3 = 1/\sqrt{3}, x_2 = -1/\sqrt{3}\).
- Fig. 12. Ruled surface \(2\Psi \) with \(x_1 = 1/\sqrt{3}, x_2 = 0, x_3 = \sqrt{2}/3\).
**Example 3:**

Considering the non-cylindrical and regular ruled surface
\[ \phi = (\varphi, \beta, \lambda) \] defined with the non-circular general helix \[ \beta = (\beta_1, \beta_2, \beta_3) \] as its base and striction curve as follows:

\[ \varphi = 2s \cos(2\ln s)/2 + \sin(2\ln s)/\sqrt{5} - 2s \cos(2\ln s)/\sqrt{5}, \]
\[ \beta_1 = 2s \sin(2\ln s)/2 - \cos(2\ln s)/\sqrt{5} - 2s \sin(2\ln s)/\sqrt{5}, \]
\[ \beta_3 = 2s/\sqrt{5} + v/\sqrt{5}. \]

Its Frenet frame \[ \{\beta(s), \varphi(s), h(s), a(s)\} \], the associated curvatures \[ k_1 \] and \[ k_2 \] and the functions \[ \alpha_1 \] and \[ \alpha_2 \] are given respectively by

\[ q = (1/\sqrt{5})(-2 \cos(2\ln s) - 2 \sin(2\ln s))1, \]
\[ h = (\sin(2\ln s)) - \cos(2\ln s),0), \]
\[ a = (1/\sqrt{5})(\cos(2\ln s), \sin(2\ln s), 2), \]
\[ k_1 = 4/(\sqrt{5}s), k_2 = 2/(\sqrt{5}s), 3 \alpha_1 = 0, 3 \alpha_2 = 1. \]

Hence, the studied family of ruled surfaces which is constructed with Frenet frame of \[ \phi \] and its characteristic properties are investigated, respectively by

\[ \Psi = (\psi_1, \psi_2, \psi_3), \]

where

\[ \psi_1 = (2s/\sqrt{5}\cos(2\ln s)/2 + \sin(2\ln s)) + \sqrt{5} + x_1 \cos(2\ln s)/\sqrt{5} + x_2 \sin(2\ln s) \]
\[ \psi_2 = (2s/\sqrt{5}\sin(2\ln s)/2 - \cos(2\ln s)) + \sqrt{5} - x_1 \cos(2\ln s)/\sqrt{5} - x_2 \sin(2\ln s) \]
\[ \psi_3 = 2s/\sqrt{5} + v/(x_1 + x_2)/\sqrt{5}. \]

\[ K(s,0) = -(4/5s^2)(2x_1^2 + x_1(2x_1 - x_3))^2/[x_1^2 + x_2^2], \]
\[ H(s,0) = -x_1 + [2x_1(2x_1^2 + x_1(2x_1 - x_3))/\sqrt{5}s(x_1^2 + x_2^2)^{1/2}]. \]

\[ \mu_1 = 2x_1/\sqrt{5}s, \]
\[ \mu_2 = 2x_2/\sqrt{5}s, \]
\[ \mu_3 = 4/(\sqrt{5}s). \]

In the following, some illustrations of some special ruled sur-
These above figures are plotted with $(s, v) = \{1/10^5, 2\pi(\pi)\} - 4.4\cdot$. 

**REFERENCES**


