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Universal quantum operators

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Abstract - In this research it is demonstrated that there is an existing universal set of elementary gates, which can emulate any other operation. The universal set of quantum operators is sufficient to satisfy all computational needs. In this formal research are reviewed the basic quantum gates and is demonstrated why they form an universal quantum gate. This particular set of operators is examined because some of the difficulty simulating quantum concepts are distributed evenly between those gates.

Index Terms— boolen function, circuit, composition, encoding, gate, quantum.

1 INTRODUCTION

In the classical computers some groups of basic operations are functionally complete. By combining and recombining such set of operations, it is possible to implement each one operation. For example the NAND gate is functionally complete in itself! Unlike the classical computers, the space of the possible quantum operations is a continuous, when we express it through discrete set of operations, the cardinals are not completely identical. However, random quantum operations can be approximated through the use of several types of gates. The quantum gates are essentially quantum operators.

In this research is determined the quantum gate acting on the k-qubit as $2^k \times 2^k$ unitary matrix, which performs a transformation of the system. When the gates are applied for a qubit system, the vector of the system state is multiplied by the matrix of the gate to obtain the new vector of the state. Since the gate is a unitary matrix, this means that the calculation is reversible - the correspondence between the inputs and outputs is one to one.

Similarly to the classic computation there is a universal set of elementary gates, which is sufficient to satisfy all computational needs [1, 2, 3, 4]. In this formal research will be reviewed the basic quantum gates and will be explained why they form an universal quantum gate. This particular set of gates will be examined because some of the difficulty simulating quantum concepts are distributed evenly between those gates. The CNOT gate controls the measurement and pairing, the Hadamard gate controls the superposition, and the phase gate controls the interference.

2 THE UNIVERSALITY

Theoretically will be proven that CNOT, Hadamard and 45° phase gate are sufficient in order to compose an universal quantum gate.

CNOT - This is a 2-qubit gate, which inverts the second qubit, if the first is set to $|1\rangle$. The CNOT gate performs the transformation CNOT $|10\rangle = |11\rangle$ and CNOT $|11\rangle = |10\rangle$.
All other states at the input remain unchanged.

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

CNOT is applied on two qubits: target and control. If the controlled qubit is 1, the target qubit changes from 1 → to 0 and vice versa. If the controlled qubit is zero, nothing changes. It is important to underline that the gate works even when the qubits are paired or in a superposition: The target qubit changes only in the parts of a superposition in which the controlled qubit is set. If CNOT is applied on q1 in q2 (this means that q1 is the controlling qubit, and q2 is the target one). If the state of the system is
$|q1 = 0, q2 = 0\rangle - |q1 = 1, q2 = 1\rangle$
after CNOT on q1 in q2 the state changes to
$|q1 = 0, q2 = 0\rangle - |q1 = 1, q2 = 0\rangle$.

The role of CNOT in this case is to express the interaction. Without it, this would be just a series of single-qubit computations. Through this gate may be implemented a number of classical simulations, regardless of whether the qubits are paired or not. Logically this gate could be simplified by replacing a CNOT gate with a CCNOT gate. This would allow all classical simulations to be implemented through one gate. The CNOT gate is beneficial also from a philosophical point of view, because the understanding of its logic is the basis of understanding the quantum measurement. The measurement of a qubit has absolutely the same effects on the quantum computations as CNOT. Whether that eliminates the problem with the measurement in the quantum mechanics or not it is controversial to say. **Toffoli** - This is a 3-qubit gate, which inverts the third qubit, if the first and second are set to $|1\rangle$. The TOFF gate performs transformation TOFF $|110\rangle = |111\rangle$ and TOFF $|111\rangle = |110\rangle$. All other states at the input remain unchanged.

$$\text{TOFF} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Hadamard - This is a 1-qubit gate, which creates a unbiased superposition of the states $|0\rangle$ and $|1\rangle$. The Hadamard gate carries out the transformations:

$$HAD|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle \frac{1}{\sqrt{2}}(|1\rangle$$
$$HAD|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - \frac{1}{\sqrt{2}}(|1\rangle$$

It should be taken into account that the Hadamard gate can be applied to higher dimensional qubit spaces by tensoring of single qubits. Hadamard gates.

$$HAD = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

If the target state of the qubit is |0⟩, the application of the Hadamard gate changes the state to |0⟩ |1⟩. If the target state of the qubit is |1⟩, the states change to |0⟩ - |1⟩. When the qubit is in a superposition, the function is applied separately to each part of a superposition.

For example when the issue for the normalizing factor $\frac{1}{\sqrt{2}}$ is put asied, in order to asses H[1] (|0⟩- |1⟩), H is applied to each part as H (|0⟩) -H (|0⟩), which extends to (| 0⟩ | 1⟩) - (| 0⟩- | 1⟩), and is simplified to 2 | 1⟩, which in fact is | 1⟩. The role of the Hadamard gate in this case is to create a superposition. Without this gate the classical simulations can not be performed because the functionally incomplete gates can not cause interference.

45° Phase gate

Similarly to Hadamard, the phase 45° gate is applied on a single qubit. But it has much more simple function: change of the relative phase between the amplitudes of the qubit states |0⟩ and |1⟩. This is achieved by keeping the state when its value is 0, and multiplication of its amplitude by $\frac{1\,i}{\sqrt{2}}$ when the value is 1. Example: We can evaluate $P_{45°}$ (|0⟩- |1⟩), through application to single components is obtained $P_{45°}$ (|0⟩) - $P_{45°}$ (|1⟩), which is estimated to be |0⟩- $\frac{1\,i}{\sqrt{2}}$|1⟩. The role of the phase 45° gate is to control the interference. The Hadamard gate creates the interference, while the phase 45° gate shall assess whether the interference is constructive, destructive or undefined. This is actually very important, because if a 45° phase gate is removed from the circuit, or is replace with the phase 90° gate, at the end, the possible classical simulation is limited to the so-called stabilizing circuits for quantum simulation.

Proof

First it is defined that the Hadamard and 45° phase gates can approximate each single qubit operation. This is in fact easily provable, since each single qubit operation corresponds to a rotation in 3D. The Hadamard operation is a rotation of 180° around the XZ axis, which is equivalent to 90° rotation around X, then around Z, then around the X axes. The phase 45° gate represents a 45° rotation around the Z axis.

The experiments carried out by the developed from the author of this report quantum simulator [10] show that the combination of these two rotations allows the performance of all other rotations. It is possible to be obtained an approximation on each target rotation, although to achieve this it may be necessary to make a composition of a longer series from the described two rotations.

On second place, the quantum operations must be factored in single qubits operations and CNOT gates. Each quantum operation corresponds to a rotation of an unitary matrix whose action is similar to the extraction of scalar values from a matrix, swapping and adding rows upon inversion with Gaussian[2] elimination. These simulations may prove to be with exponentially many factors because these matrices are with exponential size, but can always be factored. The factoring creates an exponential increase of the number of operations, but the same exponential increase occurs also at the classical circuits. Classically, these are 2^{2^N} operations, which receive the N bit at the input and return one bit at the output, and there are only ≈G^G ways to link together the G NAND gates. This exponential increase would have been even bigger upon quantum computations.

Each quantum operation can be approximately simulated by 45° phase, CNOT and Hadamard gate.

Measuring and pairing can be classically simulated through CNOT operations, creation of superposition and interference can be simulated through Hadamard operations, while the complex quantum interference can be simulated through phase gates.

With the help of the small set of quantum gates, which have been defined can be created arrays, which to be applied consistently to the quantum system. Such an array of quantum gates is also called a quantum circuit. For example, if a 2-qubit system is given and a NOT gate is applied and after that a CNOT gate, then the resulting quantum circuit is {NOT, CNOT}. Therefore, the quantum gates are the building elements of the quantum circuits. With this basis it is possible to be initiated the design of effective quantum circuits for performance of quantum walk in one and two dimensions.

3 Conclusion

Each quantum operation can be approximately emulated through 45° phase operator, CNOT tor and Hadamard operator. The measurement and the entanglement are emulated through CNOT, the superposition and interference are emulated through Hadamard operations, and the complex interference is emulated through 45° phase operator.

The universal quantum operators, can emulate any other operation.

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